Three Proofs of the Church-Rosser Theorem
in the Light of Labelled Reduction

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Abstract. A general notion of labelled reduction in the $\lambda$-calculus is considered, and a handy characterization of joint confluence for arbitrary pairs of such reductions is formulated. Since $\beta$-reduction is a special case of labelled reduction, the Church-Rosser theorem is a simple instance of this result. Other instances relate to properties of developments which have been employed in traditional proofs of the Church-Rosser theorem: the strip lemma, which the original proof is based on, and the diamond property for developments. The applicability of these proofs to $\lambda$-calculi with infinitely long terms is discussed.

1 Introduction

The Church-Rosser theorem has been a turning point in the development of the $\lambda$-calculus. Its conceptual significance was two-fold: first, it enabled the seamless interpretation of $\beta$-reduction as a notion of computation, and second, it guaranteed that the absence of obvious formal (e.g. set-theoretic) semantics for the theory was not due to some hidden paradox. Since the time of its discovery, it has been playing a central rôle in the advancement of the $\lambda$-calculus, and it has been extended to a great many functional systems and type theories.

Several proofs of the Church-Rosser theorem have been formulated over the course of the years, and they have been related to one another in various ways. While the issue seems to have been settled as far as the $\lambda$-calculus is concerned, the same is not true for some of its extensions. Further research in the direction of generalization and abstraction is required in order to understand the situation more thoroughly.

The present study, and in particular the importance of joint confluence, builds on results, contained in the author’s PhD thesis, regarding the Church-Rosser property for infinitary functional calculi. This is a considerably more difficult problem, because the traditional proof techniques do not apply directly; the interested reader may wish to consult [11] and [8] for seminal work on the subject.

The presentation is mostly self-contained. A rudimentary acquaintance with the elements of $\lambda$-calculus is assumed; good introductions are [2], [9] and [1].
2 The labelled $\lambda$-calculus

Labels were introduced in $\lambda$-calculus in the seventies as a tool for the study of semantics [12, 7] and syntax [10, 3]. The following is a general notion of labelling for the pure $\lambda$-calculus that provides a uniform treatment for many of these systems.

Terms and substitution. Given a set $L$ of labels, $L$-labelled $\lambda$-terms (or simply $\lambda^L$-terms) are defined by the mutually recursive grammar

\[
\begin{align*}
\text{pseudoterms} & : \equiv x \mid MM \mid \lambda x M, \\
\text{terms} & : \equiv P^a, \quad a \in L.
\end{align*}
\]

We reserve the letters $M$, $N$ etc. for terms and $P$, $Q$ etc. for pseudoterms. Terms are understood modulo $\alpha$-conversion, denoted $M \equiv N$. To a $\lambda^L$-term $M$ we associate a pure $\lambda$-term $\text{erase}(M)$ by stripping all labels. If $\text{erase}(M) \equiv M'$, then $M$ is an $L$-lifting of $M'$ and $M'$ is the projection of $M$.

Henceforth we shall assume that $L$ is a semigroup, i.e. it is equipped with an associative binary operation $(a,b) \mapsto ab$. Labels act on terms according to $aPb \equiv Pb$.

Substitution is defined by the usual induction, the base case being

\[x^a[x := N] \equivaN.\]

Reduction. Several relations of labelled reduction have been considered in the literature, corresponding to the various applications of labels. In order to encompass all of them, we shall adopt the following parametric description. We assume given a ternary relation $\rightarrow$ on labels, and we write $b \in c \rightarrow d$ for $(b,c,d) \in \rightarrow$. Labelled reduction is defined, modulo context, by

\[
((\lambda x M)^b N)^a \longrightarrow aM[x := cN], \quad b \in c \rightarrow d.
\]

In the above situation, $((\lambda x M)^b N)^a$ is called a redex and $aM[x := cN]$ a contractum of this redex (a redex may have several different contracta). The reflexive and transitive closure of $\longrightarrow$ is denoted $\longrightarrow$.

3 Joint confluence

Let $\rightarrow_l$ and $\rightarrow_r$ be two ternary relations on a semigroup $L$. The corresponding reduction relations $\rightarrow_l$ and $\rightarrow_r$ are jointly confluent if $\rightarrow_l$ and $\rightarrow_r$ commute: whenever $M \rightarrow_l M_l$ and $M \rightarrow_r M_r$ there is some $N$ such that $M_l \rightarrow_r N$ and $M_r \rightarrow_l N$.
and weakly jointly confluent if whenever \( M \xrightarrow{l} M_l \) and \( M \xrightarrow{r} M_r \) there is some \( N \) such that \( M_l \xrightarrow{r} N \) and \( M_r \xrightarrow{l} N \),

A reduction relation is (weakly) confluent if it is (weakly) jointly confluent with itself.

**Theorem 1** For two reduction relations \( \xrightarrow{l} \) and \( \xrightarrow{r} \) as above, the following are equivalent:

1. \( \xrightarrow{l} \) and \( \xrightarrow{r} \) are jointly confluent.
2. \( \xrightarrow{l} \) and \( \xrightarrow{r} \) are weakly jointly confluent.
3. If a redex \( R \) has an \( l \)-contractum \( C_l \) and an \( r \)-contractum \( C_r \), then \( C_l \equiv C_r \) (i.e. \( \xrightarrow{l} \) and \( \xrightarrow{r} \) agree about the contraction of common redexes).

4 **The Church-Rosser theorem**

The pure \( \lambda \)-calculus. The pure \( \lambda \)-terms can be represented in the framework of labels as \( \lambda \)-terms with one label 0. This unique label is omitted from the notation, whence the inductive definition of the terms becomes

\[
M ::= x | MM | \lambda x M.
\]

Ordinary \( \beta \)-reduction, denoted \( \xrightarrow{\beta} \), is obtained by defining \( 0 \rightarrow 0 = \{0\} \). It enjoys the following fundamental property:

**The Church-Rosser theorem.** \( \beta \)-reduction is confluent.

We shall examine three well-known proofs of this result. The first one is obtained by specializing the proof of theorem 1 to the pair \( \xrightarrow{\beta} \), \( \xrightarrow{\beta} \). This is a variant of Lévy’s proof [10] [2, 14.2.4] [1, p. 35].

In order to examine the other proofs of the Church-Rosser theorem, we need to enhance our machinery slightly. Let 2 be the two-element set \( \{0, 1\} \) equipped with the associative operation

\[
a b = \begin{cases} 
1 & \text{if } a = b = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

We define the ternary relations

\[
c \rightarrow_0 d = \begin{cases} 
\{0, 1\} & \text{if } c = d = 0, \\
\emptyset & \text{otherwise.}
\end{cases}
\]

\[
c \rightarrow_1 d = \begin{cases} 
\{1\} & \text{if } c = d = 0, \\
\emptyset & \text{otherwise.}
\end{cases}
\]

These induce relations \( \rightarrow_0 \) and \( \rightarrow_1 \) of labelled reduction between \( \lambda^2 \)-terms. For a pure \( \lambda \)-term \( M \), we write \( M^{(1)} \) for the 2-lifting obtained by attaching the label 1 to each subterm of \( M \).
Fact 2 If $M \rightarrow_0 N$, then $\text{erase}(M) \rightarrow_\beta \text{erase}(N)$.

Fact 3 If $M \rightarrow_\beta N$, then for some 2-lifting $N'$ of $N$, $M^{(1)} \rightarrow_0 N'$.

A reduction $M \rightarrow_\beta N$ is a development, denoted $M \rightarrow_d N$, if there are 2-liftings $M'$ and $N'$ of $M$ and $N$ respectively such that $M' \rightarrow_1 N'$. The double arrow merely indicates that a development is a many-step process.

Fact 4 The transitive closure of $\rightarrow_d$ is $\rightarrow_\beta$.

Fact 5 If $M \rightarrow_1 N$, then $\text{erase}(M) \rightarrow_d \text{erase}(N)$.

Fact 6 If $M \rightarrow_d N$, then for some 2-lifting $N'$ of $N$, $M^{(1)} \rightarrow_1 N'$.

Applying theorem 1 we obtain

Proposition 7 $\rightarrow_1$ and $\rightarrow_0$ are jointly confluent.

By successively lifting, completing and projecting, we arrive at

Corollary 8 $\rightarrow_d$ commutes with $\rightarrow_\beta$.

\[
\begin{array}{c}
M \\
\beta \uparrow \\
\beta \\
\downarrow \\
M_f \leftrightarrow M_r \\
\beta \\
\downarrow \\
N \\
\downarrow \\
M_t \\
\beta \\
\downarrow \\
M_f \leftrightarrow M_r \\
\beta \\
\downarrow \\
N \\
\beta \\
\downarrow \\
M_t
\end{array}
\]

It follows that the transitive closure $\rightarrow_\beta$ of $\rightarrow_d$ also commutes with $\rightarrow_\beta$, i.e. $\rightarrow_\beta$ is confluent. This proof of the Church-Rosser theorem proceeds along the lines of the original one in [4]; commutation of development with $\beta$-reduction generalizes the strip lemma [4, Lemma 2] [2, 11.1.9].

A further application of theorem 1 yields

Proposition 9 $\rightarrow_1$ is confluent.

Similarly to corollary 8 we conclude

Corollary 10 $\rightarrow_d$ commutes with itself.

\[
\begin{array}{c}
M \\
\beta \uparrow \\
\beta \\
\downarrow \\
M_f \leftrightarrow M_r \\
\beta \\
\downarrow \\
N \\
\beta \\
\downarrow \\
M_t \\
\beta \\
\downarrow \\
M_f \leftrightarrow M_r \\
\beta \\
\downarrow \\
N \\
\beta \\
\downarrow \\
M_t
\end{array}
\]

It follows that the transitive closure $\rightarrow_\beta$ of $\rightarrow_d$ also commutes with itself, i.e. $\rightarrow_\beta$ is confluent. This is apparently\(^{1}\) the proof in [5]; the crucial intermediate fact (corollary 10) is the diamond lemma for developments [2, 11.2.28(ii)].

\(^{1}\) I do not have access to [5].
5 Concluding remarks

Both the strip lemma and the diamond lemma are customarily derived from the finiteness of developments, the property that there is no infinite reduction sequence all of whose initial segments are developments. The present approach does not rely on any such termination property, and thus it also applies to infinitary systems, where developments are not necessarily finite. Unfortunately, for such systems it is no longer the case that reduction is the transitive closure of development, since reductions may involve infinitely many steps. This in turn suggests the replacement of transitive closure by a straightforward topological notion of closure. On the other hand, the direct derivation of the Church-Rosser theorem from theorem 1 entirely bypasses the issue.

The usefulness of labelled reduction is by no means limited to the study of confluence in the pure $\lambda$-calculus. Many other properties of reduction, including finiteness of developments, standardization and syntactic continuity, can be treated in a similar way. Generalized developments [6] constitute another relatively recent area of application of labels.

References