Θεωρία Τύπων

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Βασικοί τύποι

- Σύνταξη

$$
\begin{array}{rcl}
\tau & ::= & \mathrm{Int} \mid \mathrm{Bool} \\
e & ::= & n \mid \mathit{true} \mid \mathit{false} \mid e_1 + e_2 \mid e_1 \wedge e_2 \\
 & \mid & \mathrm{if}\, e \,\mathrm{then}\, e_1 \,\mathrm{else}\, e_2 \mid \ldots
\end{array}
$$

- Κανόνες τύπων

$$
\frac{}{\Gamma \vdash \mathrm{Int} : *} \qquad \frac{}{\Gamma \vdash \mathrm{Bool} : *}
$$

$$
\frac{}{\Gamma \vdash n : \mathrm{Int}} \qquad \frac{\Gamma \vdash e_1 : \mathrm{Int} \qquad \Gamma \vdash e_2 : \mathrm{Int}}{\Gamma \vdash e_1 + e_2 : \mathrm{Int}}
$$

$$
\frac{}{\Gamma \vdash \mathit{true} : \mathrm{Bool}} \qquad \frac{}{\Gamma \vdash \mathit{false} : \mathrm{Bool}}
$$

$$
\frac{\Gamma \vdash e_1 : \mathrm{Bool} \qquad \Gamma \vdash e_2 : \mathrm{Bool}}{\Gamma \vdash e_1 \wedge e_2 : \mathrm{Bool}}
$$

$$
\frac{\Gamma \vdash e : \mathrm{Bool} \qquad \Gamma \vdash e_1 : \tau \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash \mathrm{if}\, e \,\mathrm{then}\, e_1 \,\mathrm{else}\, e_2 : \tau}
$$

- Μερικοί σημασιολογικοί κανόνες

$$
\begin{array}{rcl}
1 + 2 & = & 3 \\
\mathit{true} \wedge \mathit{false} & = & \mathit{false} \\
\mathrm{if}\, \mathit{true} \,\mathrm{then}\, e_1 \,\mathrm{else}\, e_2 & = & e_1 \\
\mathrm{if}\, \mathit{false} \,\mathrm{then}\, e_1 \,\mathrm{else}\, e_2 & = & e_2
\end{array}
$$

Απλοί τύποι — (i)

- Σύνταξη

$$\begin{array}{lcl} \tau & ::= & \ldots \mid \mathrm{Unit} \mid \tau_1 \to \tau_2 \\ e & ::= & \ldots \mid \bullet \mid \lambda x : \tau .\, e \mid e_1\, e_2 \end{array}$$

- Κανόνες τύπων

$$\frac{}{\Gamma \vdash \mathrm{Unit} : *} \qquad \frac{\Gamma \vdash \tau_1 : * \qquad \Gamma \vdash \tau_2 : *}{\Gamma \vdash \tau_1 \to \tau_2 : *}$$

$$\frac{}{\Gamma \vdash \bullet : \mathrm{Unit}} \qquad \frac{\Gamma \vdash \tau : * \qquad \Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau .\, e : \tau \to \tau'}$$

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1\, e_2 : \tau'}$$

- Σημασιολογικός κανόνας

$$(\lambda x : \tau .\, e)\, e' \quad = \quad e[x := e']$$

Απλοί τύποι — (ii)

- Σύνταξη

$$\begin{array}{rcl} \tau & ::= & \ldots \mid \tau_1 \times \tau_2 \mid \tau_1 + \tau_2 \\ e & ::= & \ldots \mid \langle e_1, e_2 \rangle \mid \mathrm{fst}(e) \mid \mathrm{snd}(e) \\ & \mid & \mathrm{inl}(e) \mid \mathrm{inr}(e) \mid [e_1, e_2] \end{array}$$

- Κανόνες τύπων

$$\frac{\Gamma \vdash \tau_1 : * \qquad \Gamma \vdash \tau_2 : *}{\Gamma \vdash \tau_1 \times \tau_2 : *} \qquad \frac{\Gamma \vdash \tau_1 : * \qquad \Gamma \vdash \tau_2 : *}{\Gamma \vdash \tau_1 + \tau_2 : *}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \qquad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2}$$

$$\frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle}{\Gamma \vdash \mathrm{fst}(e) : \tau_1} \qquad \frac{\Gamma \vdash e : \langle \tau_1, \tau_2 \rangle}{\Gamma \vdash \mathrm{snd}(e) : \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_1 \qquad \Gamma \vdash \tau_2 : *}{\Gamma \vdash \mathrm{inl}(e) : \tau_1 + \tau_2} \qquad \frac{\Gamma \vdash \tau_1 : * \qquad \Gamma \vdash e : \tau_2}{\Gamma \vdash \mathrm{inr}(e) : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \to \tau \qquad \Gamma \vdash e_2 : \tau_2 \to \tau}{\Gamma \vdash [e_1, e_2] : (\tau_1 + \tau_2) \to \tau}$$

- Μερικοί σημασιολογικοί κανόνες

$$\begin{array}{rcl} \mathrm{fst}(\langle e_1, e_2 \rangle) & = & e_1 \\ \mathrm{snd}(\langle e_1, e_2 \rangle) & = & e_2 \\ [e_1, e_2]\,(\mathrm{inl}(e)) & = & e_1\, e \\ [e_1, e_2]\,(\mathrm{inr}(e)) & = & e_2\, e \end{array}$$

Αναδρομικές συναρτήσεις

- Σύνταξη

$$e \; ::= \; \ldots \; \mid \; \mathrm{fix}(e)$$

- Κανόνες τύπων

$$\frac{\Gamma \vdash e : \tau \to \tau}{\Gamma \vdash \mathrm{fix}(e) : \tau}$$

- Σημασιολογικός κανόνας

$$\mathrm{fix}(f) \quad = \quad f\,(\mathrm{fix}(f))$$

- Παράδειγμα

$$\begin{array}{lcl} p & \equiv & \mathrm{fix}(\lambda f : \mathrm{Int} \to \mathrm{Int}.\, \lambda n : \mathrm{Int}. \\ & & \qquad \mathrm{if}\, n \leq 0\, \mathrm{then}\, 1\, \mathrm{else}\, n * f\,(n-1)) \end{array}$$

$$\begin{array}{lcl} p\,3 & = & (\lambda f : \mathrm{Int} \to \mathrm{Int}.\, \lambda n : \mathrm{Int}. \\ & & \mathrm{if}\, n \leq 0\, \mathrm{then}\, 1\, \mathrm{else}\, n * f\,(n-1))\, p\,3 \\ & = & (\lambda n : \mathrm{Int}.\, \mathrm{if}\, n \leq 0\, \mathrm{then}\, 1\, \mathrm{else}\, n * p\,(n-1))\, 3 \\ & = & \mathrm{if}\, 3 \leq 0\, \mathrm{then}\, 1\, \mathrm{else}\, 3 * p\,(3-1) \\ & = & \mathrm{if}\, \mathit{false}\, \mathrm{then}\, 1\, \mathrm{else}\, 3 * p\,2 \\ & = & 3 * p\,2 \\ & = & \ldots \\ & = & 3 * (2 * (1 * 1)) \;\; = \;\; \ldots \;\; = \;\; 6 \end{array}$$

Πολυμορφισμός 2ης τάξης

- Σύνταξη

$$\begin{array}{lcl} \kappa & :: & * \\ \tau & ::= & \ldots \mid \alpha \mid \forall \alpha : \kappa.\ \tau \\ e & ::= & \ldots \mid \Lambda \alpha : \kappa.\, e \mid e\,[\tau] \end{array}$$

- Κανόνας τύπων

$$\frac{}{\Gamma \vdash * : \Box} \qquad \frac{\Gamma \vdash \kappa : \Box \qquad \Gamma, \alpha : \kappa \vdash \tau : *}{\Gamma \vdash \forall \alpha : \kappa.\ \tau : *}$$

$$\frac{\Gamma \vdash \kappa : \Box \qquad \Gamma, \alpha : \kappa \vdash e : \tau}{\Gamma \vdash \Lambda \alpha : \kappa.\, e : \forall \alpha : \kappa.\ \tau}$$

$$\frac{\Gamma \vdash e : \forall \alpha : \kappa.\ \tau}{\Gamma \vdash e\,[\tau'] : \tau[\alpha := \tau']}$$

- Σημασιολογικός κανόνας

$$(\Lambda \alpha : \kappa.\, e)\,[\tau] \quad = \quad e[\alpha := \tau]$$

- Παραδείγματα

$$\begin{array}{rcl} \mathit{id} & : & \forall \alpha : *.\ \alpha \to \alpha \\ & \equiv & \Lambda \alpha : *.\, \lambda x : \alpha.\, x \end{array}$$

$$\begin{array}{rcl} \mathit{comp} & : & \forall \alpha : *.\ \forall \beta : *.\ \forall \gamma : *. \\ & & \quad (\beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma \\ & \equiv & \Lambda \alpha : *.\, \Lambda \beta : *.\, \Lambda \gamma : *. \\ & & \quad \lambda f : \beta \to \gamma.\, \lambda g : \alpha \to \beta.\, \lambda x : \alpha.\, f\,(g\,x) \end{array}$$

Πολυμορφισμός υψηλής τάξης

- Σύνταξη

$$\begin{array}{rcl} \kappa & ::= & \ldots \mid \kappa \to \kappa' \\ \tau & ::= & \ldots \mid \lambda\alpha : \kappa .\, \tau \mid \tau\, \tau' \end{array}$$

- Κανόνες τύπων

$$\frac{\Gamma \vdash \kappa : \Box \qquad \Gamma \vdash \kappa' : \Box}{\Gamma \vdash \kappa \to \kappa' : \Box}$$

$$\frac{\Gamma \vdash \kappa : \Box \qquad \Gamma, \alpha : \kappa \vdash \tau : \kappa'}{\Gamma \vdash \lambda\alpha : \kappa .\, \tau : \kappa \to \kappa'}$$

$$\frac{\Gamma \vdash \tau_1 : \kappa \to \kappa' \qquad \Gamma \vdash \tau_2 : \kappa}{\Gamma \vdash \tau_1\, \tau_2 : \kappa'}$$

- Σημασιολογικός κανόνας

$$(\lambda\alpha : \kappa .\, \tau)\, \tau' \quad = \quad \tau[\alpha := \tau']$$

- Παράδειγμα

$$\begin{array}{rcl} f & : & \forall m : * \to * .\, (\forall \alpha : * .\, \alpha \to m\, \alpha) \to \\ & & \quad \forall \alpha : * .\, \forall \beta : * .\, (\alpha \times \beta) \to (m\, \alpha \times m\, \beta) \\ & = & \Lambda m : * \to * .\, \lambda g : (\forall \alpha : * .\, \alpha \to m\, \alpha). \\ & & \quad \Lambda \alpha : * .\, \Lambda \beta : * .\, \lambda p : \alpha \times \beta. \\ & & \qquad \langle g\, [\alpha]\, (\mathrm{fst}(p)), g\, [\beta]\, (\mathrm{snd}(p)) \rangle \end{array}$$

Υπαρξιακοί τύποι

- Σύνταξη

$$\begin{array}{lcl} \tau & ::= & \ldots \mid \exists\alpha : \kappa.\ \tau \\ e & ::= & \ldots \mid \mathrm{pack}(\alpha : \kappa = \tau, e : \tau') \\ & \mid & \mathrm{open}\, e \,\mathrm{as}\, (\alpha : \kappa, x : \tau) \,\mathrm{in}\, e' \end{array}$$

- Κανόνες τύπων

$$\frac{\Gamma \vdash \kappa : \Box \qquad \Gamma, \alpha : \kappa \vdash \tau : \kappa'}{\Gamma \vdash \exists\alpha : \kappa.\ \tau : *}$$

$$\frac{\Gamma \vdash \tau : \kappa \qquad \Gamma \vdash e : \tau'[\alpha := \tau]}{\Gamma \vdash \mathrm{pack}(\alpha : \kappa = \tau, e : \tau') : \exists\alpha : \kappa.\ \tau'}$$

$$\frac{\Gamma \vdash \tau' : * \qquad \Gamma \vdash e : \exists\alpha : \kappa.\ \tau \qquad \Gamma, \alpha : \kappa, x : \tau \vdash e' : \tau'}{\Gamma \vdash \mathrm{open}\, e \,\mathrm{as}\, (\alpha : \kappa, x : \tau) \,\mathrm{in}\, e' : \tau'}$$

- Σημασιολογικός κανόνας

$$\begin{array}{l} \mathrm{open}\, (\mathrm{pack}(\alpha : \kappa = \tau, e : \tau')) \,\mathrm{as}\, (\alpha : \kappa, x : \tau') \,\mathrm{in}\, e' \\ \quad = \ (e'[\alpha := \tau])[x := e] \end{array}$$

- Παράδειγμα

$$\begin{array}{rcl} \mathit{stream} & : & * \to * \\ & \equiv & \lambda\alpha : *. \exists\beta : *.\ (\beta \times (\beta \to \alpha)) \times (\beta \to \beta) \end{array}$$

$$\begin{array}{rcl} \mathit{elem} & : & \forall\alpha : *.\ \mathit{stream}\,\alpha \to \mathrm{Int} \to \alpha \\ & \equiv & \Lambda\alpha : *. \lambda s : \mathit{stream}\,\alpha. \\ & & \mathrm{open}\, s \,\mathrm{as}\, (\beta : *, i : (\beta \times (\beta \to \alpha)) \times (\beta \to \beta)) \,\mathrm{in} \\ & & \mathrm{fix}(\lambda f : \beta \to \mathrm{Int} \to \alpha. \lambda x : \beta. \lambda n : \mathrm{Int}. \\ & & \quad \mathrm{if}\, n \leq 0 \,\mathrm{then}\, \mathrm{snd}(\mathrm{fst}(i))\, x \\ & & \qquad\qquad \mathrm{else}\ \ f\, (\mathrm{snd}(i)\, x)\, (n - 1))\, (\mathrm{fst}(\mathrm{fst}(i))) \end{array}$$

Υποτύποι

- Σύνταξη

$$\tau \ ::= \ \ldots \ \mid \ \mathrm{Top}$$

- Σχέση υποτύπων

$$\frac{}{\Gamma \vdash \tau <: \tau} \qquad \frac{}{\Gamma \vdash \tau <: \mathrm{Top}}$$

$$\frac{\Gamma \vdash \tau_1 <: \tau_2 \qquad \Gamma \vdash \tau_2 <: \tau_3}{\Gamma \vdash \tau_1 <: \tau_3}$$

$$\frac{\Gamma \vdash \tau_1' <: \tau_1 \qquad \Gamma \vdash \tau_2 <: \tau_2'}{\Gamma \vdash \tau_1 \to \tau_2 <: \tau_1' \to \tau_2'}$$

- Κανόνας τύπων

$$\frac{\Gamma \vdash e : \tau \qquad \Gamma \vdash \tau <: \tau'}{\Gamma \vdash e : \tau'}$$

- Παράδειγμα

$$\mathrm{Nat} \ <: \ \mathrm{Int} \ <: \ \mathrm{Real}$$

Αναδρομικοί τύποι

- Σύνταξη

$$\begin{array}{rcl} \tau & ::= & \ldots \;\mid\; \mu\alpha.\,\tau \\ e & ::= & \ldots \;\mid\; \mathrm{unfold}(e) \;\mid\; \mathrm{fold}_{\mu\alpha.\,\tau}(e) \end{array}$$

- Κανόνες τύπων

$$\frac{\Gamma, \alpha : * \vdash \tau : *}{\Gamma \vdash \mu\alpha.\,\tau : *} \qquad \frac{\Gamma \vdash e : \tau[\alpha := \mu\alpha.\,\tau]}{\Gamma \vdash \mathrm{fold}_{\mu\alpha.\,\tau}(e) : \mu\alpha.\,\tau}$$

$$\frac{\Gamma \vdash e : \mu\alpha.\,\tau}{\Gamma \vdash \mathrm{unfold}(e) : \tau[\alpha := \mu\alpha.\,\tau]}$$

- Σημασιολογικός κανόνας

$$\mathrm{unfold}(\mathrm{fold}_{\mu\alpha.\,\tau}(e)) \quad = \quad e$$

- Παράδειγμα

$$\begin{array}{rcl} \mathit{list} & : & * \to * \\ & \equiv & \lambda\alpha : *.\, \mu\beta.\, \mathrm{Unit} + (\alpha \times \beta) \\[1ex] \mathit{nil} & : & \forall\alpha : *.\ \mathit{list}\,\alpha \\ & \equiv & \Lambda\alpha : *.\, \mathrm{fold}_{\mathit{list}\,\alpha}(\mathrm{inl}(\bullet)) \\[1ex] \mathit{cons} & : & \forall\alpha : *.\ \alpha \to \mathit{list}\,\alpha \to \mathit{list}\,\alpha \\ & \equiv & \Lambda\alpha : *.\, \lambda h : \alpha.\, \lambda t : \mathit{list}\,\alpha.\, \mathrm{fold}_{\mathit{list}\,\alpha}(\mathrm{inr}(\langle h, t\rangle)) \\[1ex] \mathit{length} & : & \forall\alpha : *.\ \mathit{list}\,\alpha \to \mathrm{Int} \\ & \equiv & \Lambda\alpha : *.\, \mathrm{fix}(\lambda f : \mathit{list}\,\alpha \to \mathrm{Int}.\, \lambda\ell : \mathit{list}. \\ & & \qquad [\lambda u : \mathrm{Unit}.\, 0, \lambda p : \alpha \times \mathit{list}\,\alpha.\, 1 + f\,(\mathrm{snd}(p))] \\ & & \qquad\quad (\mathrm{unfold}(\ell))) \end{array}$$

Εξαρτώμενοι τύποι

- Σύνταξη

$$\begin{array}{rcl} \kappa & ::= & \ldots \mid \tau \to \kappa \\ \tau & ::= & \ldots \mid \Pi x : \tau.\ \tau' \mid \lambda x : \tau. \tau' \mid \tau\, e \\ & \mid & \text{if}\ e\ \text{then}\ \tau_1\ \text{else}\ \tau_2 \end{array}$$

- Συντομογραφία: $\tau \to \tau' \equiv \Pi x : \tau.\ \tau'$
αν $x \notin \mathrm{FV}(\tau')$ (μη εξαρτώμενη συνάρτηση)

- Κανόνες τύπων

$$\frac{\Gamma \vdash \tau : * \quad \Gamma \vdash \kappa : \Box}{\Gamma \vdash \tau \to \kappa : \Box} \qquad \frac{\Gamma \vdash \tau : * \quad \Gamma, x : \tau \vdash \tau' : *}{\Gamma \vdash \Pi x : \tau.\ \tau' : *}$$

$$\frac{\Gamma, x : \tau \vdash \tau' : \kappa}{\Gamma \vdash \lambda x : \tau. \tau' : \tau \to \kappa}$$

$$\frac{\Gamma \vdash \tau : \tau' \to \kappa \quad \Gamma \vdash e : \tau'}{\Gamma \vdash \tau\, e : \kappa}$$

$$\frac{\Gamma, x : \tau \vdash e : \tau' \quad \Pi x : \tau.\ \tau' : *}{\Gamma \vdash \lambda x : \tau. e : \Pi x : \tau.\ \tau'}$$

$$\frac{\Gamma \vdash e : \Pi x : \tau.\ \tau' \quad \Gamma \vdash e' : \tau}{\Gamma \vdash e\, e' : \tau'[x := e']}$$

$$\frac{\Gamma \vdash e : \tau \quad \tau = \tau' \quad \Gamma \vdash \tau' : *}{\Gamma \vdash e : \tau'}$$

$$\frac{\Gamma \vdash \tau : \kappa \quad \kappa = \kappa' \quad \Gamma \vdash \kappa' : \Box}{\Gamma \vdash \tau : \kappa'}$$

$$\frac{\Gamma \vdash e : \mathrm{Bool} \quad \Gamma \vdash \tau_1 : \kappa \quad \Gamma \vdash \tau_2 : \kappa}{\Gamma \vdash \text{if}\ e\ \text{then}\ \tau_1\ \text{else}\ \tau_2 : \kappa}$$