Radiocolorings in Periodic Planar Graphs: PSPACE-Completeness and Efficient Approximations for the Optimal Range of Frequencies

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Abstract. The Frequency Assignment Problem (FAP) in radio networks is the problem of assigning frequencies to transmitters exploiting frequency reuse while keeping signal interference to acceptable levels. The FAP is usually modelled by variations of the graph coloring problem. The Radiocoloring (RC) of a graph $G(V,E)$ is an assignment function $\Phi : V \rightarrow \mathbb{N}$ such that $|\Phi(u) - \Phi(v)| \geq 2$, when u, v are neighbors in G, and $|\Phi(u) - \Phi(v)| \geq 1$ when the distance of u, v in G is two. The range of frequencies used is called *span*. Here, we consider the optimization version of the Radiocoloring Problem (RCP) of finding a radiocoloring assignment of minimum span, called *min span RCP*.

In this paper, we deal with a variation of RCP: that of satisfying frequency assignment requests with some *periodic* behavior. In this case, the interference graph is an (infinite) periodic graph. Infinite periodic graphs model finite networks that accept periodic (in time, e.g. daily) requests for frequency assignment. Alternatively, they may model very large networks produced by the repetition of a small graph.

A *periodic graph* G is defined by an infinite two-way sequence of repetitions of the same finite graph $G_i(V_i, E_i)$. The edge set of G is derived by connecting the vertices of each iteration G_i to some of the vertices of the next iteration G_{i+1}, the same for all G_i. The model of periodic graphs considered here is similar to that of periodic graphs in Orlin [13], Marathe et al [10]. We focus on planar periodic graphs, because in many cases real networks are planar and also because of their independent mathematical interest. We give two basic results:

- We prove that the *min span RCP is $\mathcal{PSPACE}$-complete for periodic planar graphs.*
- We provide an $O(n(\Delta(G_i) + \sigma))$ time algorithm, (where $|V_i| = n$, $\Delta(G_i)$ is the maximum degree of the graph G_i and σ is the number of edges connecting each G_i to G_{i+1}), which obtains a radiocoloring of a periodic planar graph G that *approximates the minimum span within a ratio which tends to 2 as $\Delta(G_i) + \sigma$ tends to infinity.*

* This work has been partially supported by the EU IST/FET projects ALCOM-FT, FLAGS, CRESCCO and EU/RTN Project ARACNE. Part of the last author's work was done during his visit at Max-Planck-Institute für Informatik (MPI).

1 Introduction, Previous Work and our Results

1.1 The Radiocoloring Problem

The Frequency Assignment Problem (FAP) in radio networks is a well-studied, interesting and well motivated problem, aiming at assigning frequencies to transmitters exploiting frequency reuse while keeping signal interference to acceptable levels. The FAP is usually modeled by variations of the graph coloring problem. The interference between transmitters is usually modelled by the interference graph $G(V, E)$, where the set V corresponds to the set of transmitters and E represents distance constraints. The set of colors represents the available frequencies. In addition, the color of each vertex in a particular assignment gets an integer value which has to satisfy certain inequalities compared to the values of colors of nearby nodes in the interference graph G (frequency-distance constraints). We study an important variation of FAP, the Radiocoloring Problem (RCP):

Definition 1. Radiocoloring Problem (RCP): *Given a graph $G(V, E)$ consider a function $\Phi : V \to N$ such that $|\Phi(u) - \Phi(v)| \geq 2$ if $d(u, v) = 1$ and $|\Phi(u) - \Phi(v)| \geq 1$ if $d(u, v) = 2$, where $d(u, v)$ is the distance between u and v in G. The function Φ is a radiocoloring (RC) of the graph G. We call the problem of finding such an assignment the Radiocoloring Problem (RCP).*

An important parameter of a radiocoloring assignment is the following:

Definition 2. span: *The number $\nu = max_{v \in V} \Phi(v) - min_{u \in V} \Phi(u) + 1$ used in a radiocoloring assignment Φ is called the span of the assignment Φ.*

The optimization version of RCP corresponding to this parameter, considered here, is the following:

Definition 3. min span RCP: *Given a graph G, find a radiocoloring assignment for the graph G of minimum span, denoted by $X_{span}(G)$.*

Note that this version of RCP has been proved to be $\mathcal{NP}$-hard even for planar and other restricted families of ordinary graphs ([5, 2]).

1.2 The min span RCP in periodic graphs

In this work we investigate the min span RCP for an interesting family of infinite planar graphs, called *periodic planar graphs*. A periodic graph G is defined by an infinite sequence of repetitions of the same finite graph $G_i(V_i, E_i)$. The edge set of G is derived by connecting some of the vertices of each iteration G_i to some of the vertices of the next iteration G_{i+1}, the same for all iterations.

Infinite periodic graphs usually represent finite networks that accept periodic (in time, e.g. daily) requests for frequency assignment. We note that periodic interference graphs usually represent networks of great practical interest, since in many networks the requests for frequency assignment exhibit some periodic behavior. That is, the network accepts periodic (e.g. daily) requests for frequency assignment. Each request has a starting and ending time and a node where it is applied. Two requests interfere if they apply for nearby nodes and their time intervals overlap. The assignment should be such that there is no time overlap between any two nearby requests of the same or the preceding and following periods of requests. Alternatively, infinite periodic graphs can model very large networks produced by the repetition of a small graph. Note in this context that many real networks consist of the repetition of the same component.

We focus here on *planar* periodic graphs, because in many cases real networks are planar and because of the independent mathematical interest of this family of graphs.

Definition 4. Linear Periodic Planar Graph G**:** *A linear periodic planar graph is defined as follows:*

Let $\widetilde{G}$ *be an arbitrary finite connected planar graph. Let* V *the vertex set of* $\widetilde{G}$*. Let also* E_0 *be the edge set of* $\widetilde{G}$*. Let* E_+ *be a specific set of ordered pairs* (u, v) *of the nodes of* $\widetilde{G}$*. Note that* E_+ *must be a set of ordered pairs of vertices whose connection according to the rule* **(c2)** *below leads to planarity preservation.*

Consider the two-way infinite sequence of graphs $\ldots, G_i, G_{i+1}, \ldots$*, where each* G_i *is isomorphic to* $\widetilde{G}$*. The infinite graph* G *is obtained from this sequence as follows:*

(a) *We assume a line (in fact, any 1-dimensional infinite simple curve) on which we select discrete points* $\ldots, i, i+1, i+2, \ldots$*, such that:*
 (a1) *Each point in the line is replaced by* $\widetilde{G}$*.*
 (a2) *Each edge* $(i, i+1)$ *in the line is replaced by* E_+*.*
 (a3) *For any finite subset of consecutive points in the line, replacing the points of the line by graphs* $\widetilde{G}$ *end the edges between them by* E_+*, the resulting graph is planar.*

(b) *The vertex set of* G *is the union of the vertex sets of the sequence* $\ldots, G_i, G_{i+1}, \ldots$*.*

(c) *The edges of* G *are* **(c1)** *The union of edge sets of the sequence of* $G_i s$ *(i.e., the edge set* E_0 *of* $\widetilde{G}$*)* **(c2)** *For each pair of adjacent copies of* $\widetilde{G}$*, call them* G_i*,* G_{i+1}*, we use the* E_+ *specification of* G *to connect the nodes of* G_i *corresponding to the first elements of the pairs in* E_+ *to the nodes of* G_{i+1}*, corresponding to the second elements of the pairs in* E_+*.*

We denote a linear periodic planar graph by $G = (\widetilde{G}(V, E_0), E_+)$*.*

For an iteration i of G, $G_i(V_i, E_i)$, let E_{i-} and E_{i+} the set of edges connecting G_i with the previous and next iterations of G, respectively. Note that both E_{i-} and E_{i+} are the same as E_+ and that G_i is isomorphic to $\widetilde{G}$.

Note that this model of infinite periodic graphs is similar to the model of Orlin [13] for *l-dimensional periodic graphs* with $l = 1$. Our model is in fact equivalent to 1-level-restricted periodic graphs in Marathe et al [10], which is a special case of [13]. We consider the planar case of such graphs.

Definition 5. $(\widetilde{G}, E_+)$*: The pair* $\widetilde{G}, E_+$ *is called the finite specification of* G*.*

Observe that there can be infinite periodic graphs which are not linear. For example, the periodic graph whose graph $\widetilde{G}$ in its finite specification, is a cycle connecting vertices a, b, c, d, and $E_+ = \{(a, a), (b, b), (c, c), (d, d)\}$, is not a linear periodic planar graph, since there is no line of discrete points such that each point in the line can be replaced by $\widetilde{G}$ and each edge of it can be replaced by E_+ leading to planarity.

Note 1. All our results refer to linear periodic planar graphs, which we call periodic planar graphs in the sequel.

1.3 Our results

1. We first prove that *the min span RCP is $\mathcal{PSPACE}$-complete for periodic planar graphs.* (The space is polynomial with respect to the size of the finite specification $\widetilde{G}$.)
2. We provide an $O(n(\Delta(G_i)+\sigma))$ time algorithm, (where $|V_i| = n$, $\Delta(G_i)$ is the maximum degree of the graph G_i and σ is the number of edges connecting G_i to G_{i+1}), which obtains a radiocoloring of a periodic planar graph G that *approximates the minimum span within a ratio which tends to* 2 *as* $\Delta(G_i)+\sigma$ *tends to infinity.*
We remark that, any approximation algorithm for the min span RCP of a finite planar graph G, that achieves a span of at most $\alpha\Delta(G) + constant$, for any α and where $\Delta(G)$ is the maximum degree of G, can be used as a subroutine in our algorithm to produce an approximation for min span periodic planar RCP of asymptotic ratio α for periodic planar graphs.

1.4 Previous Work

The FAP has been considered in e.g. [4, 1]. The problem of min span RCP for ordinary planar graphs was proved to be $\mathcal{NP}$-complete in [5, 6, 2]. In [5, 1] the authors provide an efficient approximation algorithm for the min order RCP of planar graphs achieving an approximation ratio tending to 2 and 1.8 (for graphs of large degree ($\Delta(G) \geq 749$)), respectively. The work of [2] presents efficient approximations for some interesting families of graphs: outerplanar, bounded treewidth, permutation and split graphs.

A model for periodic graphs (called *l-dimensional periodic graphs*) was first presented by Orlin in [13]. The model of periodic graphs considered in this work is similar to that of Orlin for the 1-dimensional case, $l = 1$ (also called *1-dimensional periodically specified graphs* or simply *periodically specified graphs*), when restricted to planar instances.

The complexity of various basic problems of periodically specified graphs was studied by Orlin [13] and Wanke [17]. In [13, 11, 16] it is proved that the problems of Maximum Independent Set (MIS), Hamiltonial Path, Partition into Triangles, SAT, 3-coloring for periodically specified graphs are $\mathcal{PSPACE}$-complete. The appoximability of basic problems on infinite periodic graphs was studied by several researchers ([3, 8, 14]) giving efficient algorithms for solving problems such as determining strongly connected components, testing the existence of cycles, bipartiteness, planarity and minimum cost spanning forests for periodically specified graphs.

Marathe et al [10] presented several $\mathcal{PSPACE}$-hardness results and also efficient approximation schemes for partitioning problems including MIS, min vertex cover and max-SAT for periodically specified graphs when restricted to planar instances. However, their approximation technique for periodically specified graphs (illustrated for the MIS problem) can not directly apply for coloring problems, considered here, because it takes the union of partial solutions-subsets of the infinite graph and thus it does not consider all the vertices; something not allowed in coloring problems.

Because of the page limit, we have included full proofs of some Lemmas and Theorems in the full version of the paper [7].

2 Embeddings of Periodic Planar Graphs

In this paper, we use the notion of an *embedding* of a planar graph.

Definition 6. Planar Embedding (of a periodic graph G**)** *([12]): For each node v of G, there is an adjacency list, such that all neighbours of v appear in clockwise order with respect to an actual drawing of G.*

The following Lemma reveals important information about the structure of a linear periodic planar graph.

Lemma 1. *Any linear periodic planar graph G can be embedded in the plane by interchanging at most two different planar embeddings of the graph obtained by an iteration i of G, G_i (which is isomorphic to $\widetilde{G}$) and the set of edges connecting G_i with the previous and next iterations, sets E_{i-}, E_{i+} (each of which is equal to E_+), called* Extended $\widetilde{G}$.

Proof. Observe first that there are cases where we need to interchange two different embeddings of the graph Extended $\widetilde{G}$ in order to draw a linear periodic planar graph preserving planarity. As an example, consider the periodic graph whose graph $\widetilde{G}$ in its finite specification is a single edge connecting two vertices a, b, and $E_+ = \{(a,b),(b,a)\}$. See the graph of Figure 1(a). We will show that interchanging at most two different embeddings of the graph Extended $\widetilde{G}$ is enough to draw any linear periodic planar graph G preserving planarity.

In order to check whether it is possible that three embeddings to be required, we need to consider any 3 consecutive iterations of G. Recall that different embeddings may be introduced because of the connections between nodes of consecutive iterations. Hence, only nodes of the exterior face of each iteration will be involved. So, we can view the three consecutive iterations as three cycles. Moreover, we can consider three simple lines, since less edges are involved in the embeddings of consecutive iterations in the case of lines compared to cycles.

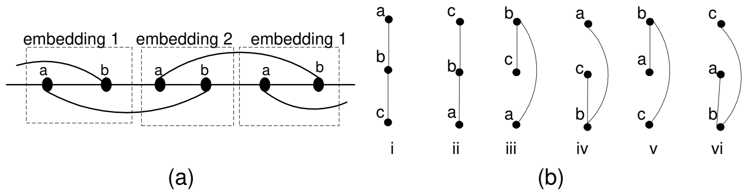

Fig. 1. (a) For this graph we need to interchange two different embeddings of the graph Extended $\widetilde{G}$, in order to draw the periodic graph in the plane without crossings. (b) The six possible 'drawings' of Line L_3.

Finally, observe that, in order for three or more planar embeddings to be needed, we have to consider at least 3 nodes of G_i, assume a, b, c. So, we can consider a line consisting of nodes a, b, c, having edges ab, bc (denoted by L_3).

We distinguish 6 different 'drawings' of a line L_3 in the plane (see Figure 1(b)). Note that some of them are equivalent embeddings with respect to definition 6 of a planar embedding. However, we consider all 6 of them since for an iteration i each such drawing combined with sets of edges E_{i-} and E_{i+} can result in a distinct embedding of the graph Extended $\widetilde{G}$. In our case the Extended $\widetilde{G}$ is the graph obtained by L_3 of iteration i and a set of edges connecting L_3 with the previous and next iterations, (sets E_{i-}, E_{i+}). In the following, we use the term 'drawing' when we refer to any of these 6 drawings and the term 'embedding' when we refer to any of them together with an instance of sets E_{i-} and E_{i+}.

To check whether three embeddings may be needed, we need to check all possible triples of these drawings. For each such triple, we check all possible sets E_+ that can lead to a linear periodic graph. For each such set, we show that interchanging at most two of these six drawings in any two consecutive iterations, is enough to draw the infinite graph in the plane. Recall that two drawings (of consecutive iterations i and $i+1$) combined with the sets of edges connecting each iteration with next and previous iterations (sets E_{i-}, E_{i+} and $E_{(i+1)-}$, $E_{(i+1)+}$) result in two different embeddings of the graph Extended $\widetilde{G}$. Henceforth, interchanging two different embeddings of the graph Extended $\widetilde{G}$ we can draw the infinite periodic planar graph in the plane without crossings.

By exhaustive check of all possible cases (see the full version [7]), we conclude that in all cases, interchanging at most two embeddings of the graph $\widetilde{G}$, is enough to draw the linear periodic graph in the plane, without edge crossings. □

Lemma 2. *Assume that we construct an infinite graph as in Definition 4 except that instead of* **(a3)** *we have that for any three consecutive points of the line, replacing each point by graph* $\widetilde{G}$ *and the edges among consecutive points by* E_+*, the resulting graph is planar.*

Then, the infinite graph thus constructed is a linear periodic planar graph.

Proof. The above requirement implies **(a3)** of Def. 4 because of Lemma 1. To see why, observe that there are two possible ways to draw a linear periodic planar graph: using the infinite sequence $\ldots, A, B, A, \ldots$ or $\ldots, B, A, B, \ldots$ (where A, B are the two planar embeddings of $\widetilde{G}$ needed). Our assumption guarantees that the coexistence of AB and BA graphs (together with E_+) has a planar embedding. Thus, by induction, any longer string will also have a planar embedding. $\square$

3 The $\mathcal{PSPACE}$-Completeness of min span RCP for Periodic Planar Graphs

We prove have that min span RCP is $\mathcal{PSPACE}$-complete for periodic planar graphs. In order to show this, we need to prove that a number of problems are $\mathcal{PSPACE}$-complete. Our $\mathcal{PSPACE}$-completeness proofs utilize known constructions for the $\mathcal{NP}$-completeness of corresponding problems for ordinary graphs. However, note that applying those constructions on the infinite periodic graph G we need infinite time to get the transformed graph G' with the desired properties. We manage to apply the transformation only to a part of the infinite graph (an iteration) and from the obtained graph to get the transformation of the whole infinite periodic graph, thus the new graph G', in time polynomial to the size of the finite specification of G. Moreover, the new graph fulfills the desired properties. These are achieved by exploiting some 'locality characteristics' that the constructions utilized here exhibit, i.e. the construction applies locally on a part (vertex or edge) of this part involving only information of the neighborhood of the part and it affects only to this neighborhood. This, combined with the repetitive structure of an infinite periodic graph enables us to get constructions of polynomial time in the size of the finite specification of the graph.

Let any iteration $G_i(V_i, E_i)$ of G and let E_{i+} the set of edges connecting any iteration G_i to the next iteration G_{i+1}. A *constant period 4-edge coloring* of a periodic graph G is a 4-edge coloring of G that assigns to each edge uv of E_i and E_{i+} of any iteration G_i the same color as the color assigned to the corresponding edge $u'v'$ of any other iteration G_j of G.

Lemma 3. *The problem of deciding whether a periodic planar graph* $G = (\widetilde{G}(V, E_0), E_+)$ *is 3-colorable (also called periodic planar 3-coloring) is* $\mathcal{PSPACE}$*-complete.*

Proof. (a) *Membership in* $\mathcal{PSPACE}$*:*
The proof that this problem is in $\mathcal{PSPACE}$ is similar to that for periodic SAT [16]: The 3-coloring of two adjacent iterations must repeat after at most 3^{2n} copies, and thus the whole 3-coloring can be assumed to be periodic with exponential period. A polynomial-space machine can then guess and check such a 3-coloring.

More analytically, suppose that the given periodic graph is 3-colorable, and consider a valid 3-coloring of it. This assignment consists of a two-way infinite $\ldots, T_i, T_{i+1}, T_{i+2}, \ldots$ of valid 3-coloring assignments to the various blocks of nodes (one for each iteration). Each T_i is an element of $\{1, 2, 3\}^n$, where $n = |V|$ is the number of vertices of the graph G_i.

The ith *chunk*, where i is any integer, is the pair (T_i, T_{i+1}), of two consecutive valid 3-coloring assignments. Since there are 3^{2n} possible different chunks, there must be two chunks, not further than 3^{2n} from each other, that are identical. That is $(T_i, T_{i+1}) = (T_j, T_{j+1})$ for some i and j between $i+2$ and $i+3^{2n}$. But this means that there is a 3-coloring assignment consisting of a two-way infinite repetition of $(T_i, T_{i+1}, \ldots, T_{j-1})$. We conclude that *if a periodic graph G is 3-colorable, then it has a periodic 3-coloring assignment with period at most exponential in the number of nodes of one iteration.*

This crucial observation allows us to solve the problem in polynomial space: Using non-determinism we can guess valid assignments $T_1, T_2, \ldots,$ always remembering the last two. After we guess T_i we check that all nodes in iteration $i-1$ are still properly colored. Once we have successfully guessed $T_{3^{2n}+2}$ we accept: We know that there is a periodic 3-coloring assignment on G.

(b) *The $\mathcal{PSPACE}$-completeness proof:*

In order to show the completeness, we reduce from 3-coloring of periodic general graphs, which is known to be $\mathcal{PSPACE}$-complete ([16]). We adapt the construction used by [15] to prove the $\mathcal{NP}$-completeness of PLANAR-3-COLORING from 3-COLORING. See the full version [7] for a complete proof. □

Lemma 4. *The problem of deciding whether a given periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$ of maximal degree four is 3-colorable is $\mathcal{PSPACE}$-complete.*

Proof. (outline) We reduce planar 3-coloring of a periodic graph with a maximum degree 4 from periodic planar 3-coloring which was proved to be $\mathcal{PSPACE}$-complete in Lemma 3 adapting the same technique used to prove that the problem of 3-coloring an ordinary planar graph of maximum degree four it is $\mathcal{NP}$-complete of [15]. *For the full proof, see [7].* □

Lemma 5. *The problem of 3-Coloring a periodic planar graph with a given constant period 4-edge coloring is $\mathcal{PSPACE}$-complete.*

Proof. (outline) We use a transformation from the 3-coloring for planar periodic graphs with maximum degree four which was proved to be $\mathcal{PSPACE}$-complete in Lemma 4 adapting the reduction used in [15] to show $\mathcal{NP}$-hardness of 3-coloring of planar graphs with maximum degree four. *For the full proof, see [7].* □

Next, we prove the main Theorem using the above results. We will show that it is $\mathcal{PSPACE}$-complete to decide whether $X_{span}(G) \leq 9$ for a given periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$.

Theorem 1. *The problem of deciding whether a periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$, of maximum degree seven, whose graph $\widetilde{G}$ in its finite specification $(\widetilde{G}, E_+)$ is a planar bipartite graph, can be radiocolored using a span of size at most 9, is $\mathcal{PSPACE}$-complete.*

Proof. We prove that the min span radiocoloring for periodic planar graphs is $\mathcal{PSPACE}$-complete, by reducing it from 3-coloring of periodic planar graphs with a given constant period 4-edge coloring which was proved to be $\mathcal{PSPACE}$-complete in Lemma 5. Our reduction adapts the construction by Bodlaender et al in [2] to show $\mathcal{NP}$-hardness of min span RCP of ordinary planar graphs reducing it from the 3-coloring of planar graphs with a given 4-edge coloring. *For the full proof of the Theorem see [7].* $\square$

As in [2], it is possible to generalize the result as follows:

Theorem 2. *Let $r \geq 8$ be an even integer. The problem of deciding whether a periodic planar graph $G = (\widetilde{G}(V, E_0), E_+)$ of maximal degree $r-2$ can be radiocolored using a span of size at most r is $\mathcal{PSPACE}$-complete.*

4 An Efficient, Constant Ratio Approximation Algorithm for min span RCP for Periodic Planar Graphs

We present an efficient time, constant ratio approximation algorithm that approximates the min span radiocoloring problem for periodic planar graphs with the same ratio as the ratio obtained by the best known approximation algorithm for planar graphs (which we use as a subroutine for the finite specification), for the same problem.

4.1 The modified graph

Consider the following partition of the periodic graph G: group together every four consecutive iterations of the graph, call the $j-th$ such group $G_{group\ j}$. More specifically,

$$G_{group\ j} = \{G(i) \bigcup G(i+1) \bigcup G(i+2) \bigcup G(i+3)\},\ \forall\ i = \ldots, -7, -3, 1, 5, 9, \ldots$$

where $j = 1, 2, 3, 4, \ldots$, (respectively, i.e. $j = \lceil i/4 \rceil$). Consider the graph $G_{group\ j}$ of G. Denote the first graph of the group as $G_{(j)1}$ or G_1, the second as $G_{(j)2}$ or G_2 and so on until the fourth. The algorithm produces a new graph, called $G'_{group\ j}$ defined as follows: The graph has the same vertex and edge set as the graph $G_{group\ j}$ except from the following modifications on the first and the fourth graphs of group $G_{group\ j}$:

Consider an edge $uv \in E_{4+}$ of graph G_4. Recall that $u \in V_4$ and the vertex v belongs to the next iteration of G, that is $v \in V_{(j*4)+1}$. For each such edge uv of the graph G_4 we do the following:

- Delete edge uv.
- Add a new edge uv' connecting the vertex $u \in G_4$ to vertex v', where $v' \in V_1$ is the corresponding to v ($v \in V_{(j*4)+1}$) vertex in graph G_1. Recall that the graphs $G_1, V_{(j*4)+1}$ are isomorphic.
- Delete edge $u''v'$ of graph G_1, where $v' \in G_1$ and u'' is the corresponding to u vertex ($u \in V_4$) in iteration G_{j*4-1} of G.

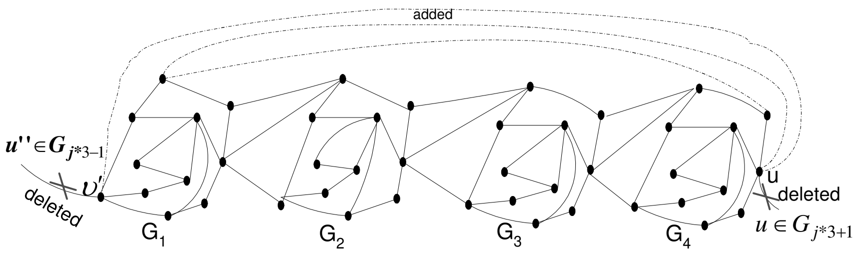

Fig. 2. The graph $G'_{group\ j}$ produced by the *Group j* of the periodic graph G

An example of the graph obtained by a periodic graph is illustrated in Figure 2. The graph $G'_{group\ j}$ has two critical properties compared to the initial periodic planar graph G: (i) it has the same maximum degree as the initial graph G, i.e. $\Delta(G'_{group\ j}) = \Delta(G)$ and (ii) as the next Lemma proves, it is a planar graph.

Lemma 6. *The modified graph $G'_{group\ j}$ is a planar graph.*

Proof. See the full version of this paper [7]. □

4.2 The Periodic Radiocoloring Partitioning Algorithm (PRPA)

The following definition is needed by the Algorithm. The definition uses the observation that the optimal span, S^*, of a radiocoloring of a graph G with maximum degree $\Delta(G)$ is clearly $S^* \geq \Delta(G)$.

Definition 7. RC Algorithm: *Let an RC Algorithm be any known min span radiocoloring polynomial time approximation algorithm for finite planar graphs with performance ratio R (when $\Delta(G)$ is used as a lower bound), i.e. if S^* is the optimal span and S_{RC} is the span obtained by the algorithm then there are constants $R > 1$ and b such that*

$$\Delta(G) \leq S^* \leq S_{RC} \leq R \cdot \Delta(G) + b$$

For example the algorithm of [9] is an RC algorithm with $R = 2$ and $b = 35$.

Algorithm PRPA*:*

1. Run an RC algorithm, on graph $G'_{group\ j}$.
   Let S_{RC} be the span obtained by RC on $G'_{group\ j}$.

2. For all $j = 1, 2, \ldots$ color the four graphs $G_{(j-1)*4+1}$, $G_{(j-1)*4+2}$, $G_{(j-1)*4+3}$, , $G_{(j-1)*4+4}$ of the group $G_{group\ j}$ as follows: Set the color of each vertex of graph $G_{(j-1)*4+k}$, $k = 1, 2, 3, 4$ to the color of its corresponding vertex, in V_k of $V(G'_{group\ j})$.
   Step 2 produces a radiocoloring of the whole periodic graph G with span S_{RC}.

Theorem 3. (**Correctness**) *The algorithm $PRPA$ produces a radiocoloring of a periodic graph G.*

Proof. (outline) It can be proved that there is no conflict either between the colors of vertices inside a group $G_{group\ j}$ or between the colors of vertices in neighbour groups. *For the full proof, see [7].* □

Theorem 4. *The Algorithm $PRPA$ runs in time $O(T(RC))$ and approximates the span within an asymptotic ratio of R, where $T(RC)$ is the time needed for the RC Algorithm to run on $G'_{group\ j}$.*

Proof. Let $\Delta(G)$ the maximum degree of the periodic graph G. Recall that the graph $G'_{group\ j}$ has maximum degree $\Delta(G'_{group\ j}) = \Delta(G)$. Denote by S the span of graph $G'_{group\ j}$ obtained by our algorithm and by $S^*(G'_{group\ j})$ the optimal span of $G'_{group\ j}$. Note that, (a) $S^* \leq S$.

(b) Also that, $S \leq R \cdot \Delta(G'_{group\ j}) + b$, by the definition of the RC Algorithm.

(c) Since, $\Delta(G'_{group\ j}) = \Delta(G)$, we get $S^* \leq S \leq R \cdot \Delta(G) + b$

Also, since $S^* \geq \Delta(G)$, we get that, $1 \leq \frac{S}{S^*} \leq R + \frac{b}{\Delta(G)}$

Finally, since $\Delta(G) \geq \Delta(\widetilde{G})$, we have $1 \leq \frac{S}{S^*} \leq R + \frac{b}{\Delta(\widetilde{G})}$.

Also, the algorithm, clearly, needs $O(T(RC))$ time, where $T(RC)$ is the time needed for algorithm RC to run on $G'_{group\ j}$. □

Remark. If the RC Algorithm is the algorithm in [9], then algorithm $PRPA$ has $R = 2$ and $b = 35$ and runs in $O(n(\Delta(G_i) + \sigma))$ time, where $n = |V_i|$, and σ is the number of edges with which each G_i is connected to G_{i+1} and $\Delta(G_i)$ is the maximum degree of G_i.

The *min order RCP* is the problem of finding a radiocoloring assignment that uses a minimum number of distinct frequencies. We conjecture that min order is $\mathcal{PSPACE}$-complete for periodic planar graphs. Note that the following modification of $PRPA$ works also for min order RCP: In Step 1, apply a known min order radiocoloring algorithm (instead of a min span RC algorithm), e.g. [5, 1], for the radiocoloring of $G'_{group\ j}$. The same analysis gives that the modified $PRPA$ algorithm approximates the min order RCP of G within an asymptotic approximation ratio of 2.

Lemma 7. *Any approximation algorithm for the min span RCP of a finite planar graph G, that achieves a span of at most $\alpha\Delta(G)$ + constant, for any α, can be used as a subroutine in algorithm $PRPA$ to produce an approximation algorithm for min span periodic planar RCP of asymptotic ratio α.*

Proof. *Using the same arguments as in Theorem 4.*

□

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