NP-completeness Results and Efficient Approximations for Radiocoloring in Planar Graphs

D.A. Fotakis, S.E. Nikoletseas, V.G. Papadopoulou, and P.G. Spirakis *

Computer Technology Institute (CTI) and Patras University, Greece
Riga Fereou 61, 26221 Patras, Greece

Abstract. The Frequency Assignment Problem (FAP) in radio networks is the problem of assigning frequencies to transmitters exploiting frequency reuse while keeping signal interference to acceptable levels. The FAP is usually modelled by variations of the graph coloring problem. The Radiocoloring (RC) of a graph $G(V, E)$ is an assignment function $\Phi : V \to \mathbb{N}$ such that $|\Phi(u) - \Phi(v)| \geq 2$, when u, v are neighbors in G, and $|\Phi(u) - \Phi(v)| \geq 1$ when the minimum distance of u, v in G is two. The discrete number and the range of frequencies used are called order and span, respectively. The optimization versions of the Radiocoloring Problem (RCP) are to minimize the span or the order. In this paper we prove that *the min span RCP is NP-complete for planar graphs*. Next, we provide an $O(n\Delta)$ time algorithm ($|V| = n$) which obtains a radiocoloring of a planar graph G that *approximates the minimum order within a ratio which tends to 2* (where Δ the maximum degree of G). Finally, we provide a *fully polynomial randomized approximation scheme* (fpras) for the *number of valid radiocolorings of a planar graph* G with λ colors, in the case $\lambda \geq 4\Delta + 50$.

1 Introduction

The Problem of Frequency Assignment in radio networks (FAP) is a well-studied, interesting problem. It is usually modelled by variations of graph coloring. The interference between transmitters are usually modelled by the interference graph $G(V, E)$, where the set V corresponds to the set of transmitters and E represents distance constraints. The set of colors represents the available frequencies. In addition, the color of each vertex in a particular assignment gets an integer value which has to satisfy certain inequalities compared to the values of colors of nearby nodes in the interference graph G (frequency-distance constraints). We here study a variation of FAP, called the Radiocoloring Problem (RCP), that models co-channel and adjacent interference constraints.

* This research is partially supported by the European Union Fifth Framework Programme Projects ALCOM-FT, ARACNE and the Greek GSRT PENED'99 Project ALKAD.

Definition 1. *(***Radiocoloring***) Given a graph $G(V, E)$ consider a function $\Phi : V \rightarrow N^*$ such that $|\Phi(u) - \Phi(v)| \geq 2$ if $D(u, v) = 1$ and $|\Phi(u) - \Phi(v)| \geq 1$ if $D(u, v) = 2$. The least possible number of colors (order) that can be used to radiocolor G is denoted by $X_{order}(G)$. The number $\nu = max_{v \in V} \Phi(v) - min_{u \in V} \Phi(u) + 1$ is called span of the radiocoloring of G and the least such number is denoted as $X_{span}(G)$.*

In most real life cases the network topology formed has some special properties, e.g. G is a lattice network or a planar graph. We remark that although there are some papers on the problem of Radiocoloring for general graphs [5, 7], there are only few works for Radiocoloring of graphs with some special characteristics such as the planar graphs.

Real networks usually reserve bandwidth (range of frequencies) rather than distinct frequencies. The objective of an assignment here is to minimize the bandwidth (span) used. The optimization version of RCP related to this objective, called *min span RCP*, tries *to find a radiocoloring for G of minimum span, $X_{span}(G)$*. However, there are cases where the objective is to minimize the distinct number of frequencies used so that unused frequencies are available for other use by the system. The related optimization version of RCP here, called *min order RCP*, tries to find *a radiocoloring that uses the minimum number of distinct frequencies, $X_{order}(G)$*. The *min span order RCP* tries to find one from all minimum span assignments that uses the minimum number of colors. The min order span RCP is defined similarly, by interchanging span to order.

Another variation of FAP considers the square of a given a graph $G(V, E)$, G^2. This is the graph of the same vertex set V and an edge set $E' : \{u, v\} \in E'$ iff $D(u, v) \leq 2$ in G. The problem is to color the square, G^2 of a graph G with the minimum number of colors, denoted as $\chi(G^2)$. This problem is studied in [12] named as *Distance-2-Coloring* (D2C). Observe that for any graph G, the minimum order of the min order RCP of G, $X_{order}(G)$, is the same as the (vertex) chromatic number of G^2, i.e. $X_{order}(G) = \chi(G^2)$. However, notice that, the set of colors used in the computed assignments of the two problems are different. The colors of the distance one vertices in the RCP should be at frequency distance two instead of one in the coloring of the G^2. However, from a valid coloring of G^2 we can always reach a valid RC of G by doubling the assigned color of each vertex. Observe also that $\chi(G^2) \leq X_{span} \leq 2\chi(G^2)$.

In [12] it is proved that the distance-2-coloring (D2C) for planar graphs is NP-complete. We remark that D2C problem is different from min span order RCP. To see this, recall that in RCP the span and order are different metrics since the distance one and two constraints are different in RCP. Note also, that the objectives of D2C and min span order RCP are different (the order and the span, respectively). Therefore, the minimum order of D2C for a given G may be different (smaller) than the minimum order of the min span order RCP of G. In [6], both by combinatorial arguments and exhaustive search, it is proved that the two problems are different. As an example, see the instance of figure 1. The minimum order of D2C for this graph is 6, while the minimum order of min span order RCP of G is 8. Note finally, that since D2C is equivalent to min order

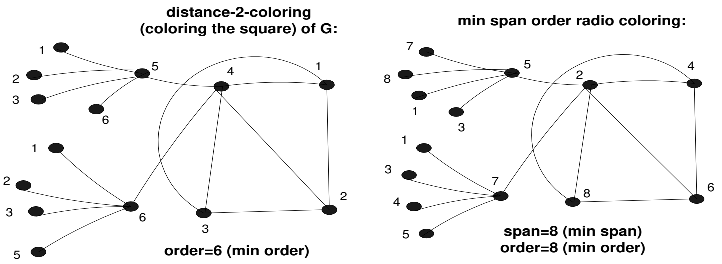

Fig. 1. An instance of G where the minimum order of distance-2-coloring and min span order RCP are different (equal to 6 and 8 respectively)

span RCP as far as the order is concerned, the min order RCP also computes a different order from the order computed in min span order RCP.

Thus, the NP-completeness of distance-2-coloring certainly does not imply the NP-completeness of min span order RCP which is here proved to be NP-complete. Additionally, the NP-completeness proof of [12] does not work for planar graphs of maximum degree $\Delta > 7$. Hence, its proof gives no information on the complexity of distance-2-coloring of planar graphs of maximum degree > 7. In contrast, our NP-completeness proof works for planar graphs of all degrees. In [12, 11] a 9-approximation algorithm for the distance-2-coloring of planar graphs is presented.

In [7] it has been proved that the problem of radiocoloring is NP-complete, even for graphs of diameter 2. The reductions use highly non-planar graphs.

In [3] a similar problem for *planar* graphs has been considered. This is the *Hidden Terminal Interference Avoidance (HTIA)* problem, which requests to color the vertices of a planar graph G so that vertices *at minimum distance exactly 2* get different colors. In [3] this problem is shown to be NP-complete.

However, the above mentioned result does not imply the NP-hardness of the min span order RCP which is proved here to be NP-complete. This so because HTIA is a different problem from RCP; in HTIA it is allowed to color neighbors in G with the same color while in RCP the colors of neighbor vertices should be at frequency distance at least two apart. Thus, the minimum number of colors as well as span needed for HTIA can vary arbitrarily from the $X_{order}(G)$ and $X_{span}(G)$. To see this consider e.g. the t-size clique graph K_t. In HTIA this can be colored with only one color. In our case (RCP) we need t colors and span of size $2t$ for K_t. In addition, the reduction used by [3], heavily exploits the fact that neighbors in G *get the same color* in the component substitution part of the reduction. Consequently, their reduction considers a different problem and it cannot be easily modified to produce an NP-hardness proof of RCP.

In this paper we are interested in *min span RCP* and in *min order RCP* of a *planar* graph G.

(a) We show that *the problem of min span RCP is NP-complete* for planar graphs. As we argued before, this result is *not* implied by the NP-completeness results of similar problems (distance-2-coloring or HTIA) [3, 12].

(b) We then present an $O(n\Delta)$ algorithm that *approximates* the minimum order of RCP, X_{order} of a planar graph G *by a constant ratio which tends to 2* as the maximum degree of G increases. Our algorithm is motivated by a constructive coloring theorem presented by Heuvel and McGuiness ([8]). Their construction can easily lead (as we show) to an $O(n^2)$ technique assuming that a planar embedding of G is given. We improve the time complexity of the approximation, and we present a much more simple algorithm to verify and implement. Our algorithm does not need any planar embedding as input.

(c) We also study the problem of *estimating the number of different radiocolorings* of a planar graph G. This is a #P-complete problem (as can be easily seen from our completeness reduction that can be done parsimonious). We use here standard techniques of rapidly mixing Markov Chains and the *newer method of coupling* for proving *rapid convergence* (see e.g. [9]) and we present *a fully polynomial randomised approximation scheme* for estimating the number of radiocolorings with λ colors for a planar graph G, when $\lambda \geq 4\Delta + 50$.

2 The NP-Completeness of the RCP for Planar Graphs

In this section, we show that the decision version of min span radiocoloring remains NP-complete for planar graphs. The decision version of min span radio coloring is, given planar graph G and an integer B, to decide whether there exists a valid radiocoloring for G of span no more than B. Therefore, the optimization version of min span radiocoloring, that is to compute a valid radiocoloring of minimum span, remains NP-hard for planar graphs.

Theorem 1. *The min span radiocoloring problem is NP-complete for planar graphs.*

Proof. The decision version of min span radio coloring clearly belongs to the class NP. To prove the theorem, we transform PLANAR-3-COLORING to min span radiocoloring. The PLANAR-3-COLORING problem is, given a planar graph $G(V, E)$, to determine whether the vertices of G can be colored with three colors, such that no adjacent vertices get the same color.

We consider a plane embedding of $G(V, E)$. Let $F(G)$ be the set of faces of G, and Δ_G be the maximum degree of G. Also, for a face $f \in F(G)$, let $size(f)$ be the number of edges of f. We define an integer Γ as $\Gamma = \max\{\Delta_G, \max_{f \in F(G)}\{size(f)\}\}$. Then, given a plane embedding of $G(V, E)$, we construct in polynomial time another planar graph $G'(V', E')$, such that there exists a radiocoloring for G' of span no more than $\Gamma + 5$ iff G is 3-colorable.

The graph $G'(V', E')$ has four kinds of vertices and three kinds of edges. As for the vertices, V' is the union of the following sets:

1. The vertex set V of the original graph G. These vertices are also called *existing* vertices, and the corresponding set is denoted by $V_E = V$.
2. The set of *intermediate* vertices V_I. There exists one intermediate vertex i_{uv} for each edge (u, v) of the original graph G.

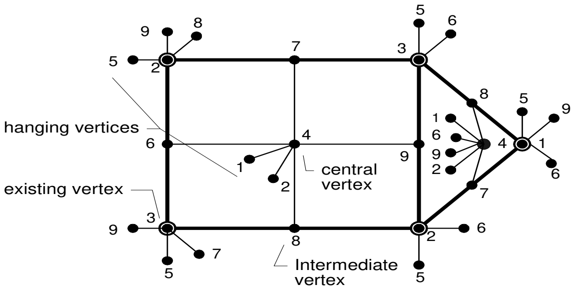

Fig. 2. The graph G' obtained from an instance G of the planar 3-coloring problem ($\Gamma = 4$)

3. The set of *central* vertices V_C. There exists one central vertex w_f for each face $f \in F(G)$.
4. The set of *hanging* vertices V_H. The hanging vertices are used to increase the degree of the existing and the central vertices to $\Gamma+1$ and $\Gamma+2$ respectively. For each $w_e \in V_E$ (resp. $w_c \in V_C$), an appropriate set of vertices $V_H(w_e)$ (resp. $V_H(w_c)$) is added so that $deg_{G'}(w_e) = \Gamma+1$ (resp. $deg_{G'}(w_c) = \Gamma+2$).

Additionally, G' contains the following sets of edges:

- The set E_I replacing the edge set E of the original graph G. In particular, we replace each $(u,v) \in E$ with two edges (u, i_{uv}) and (i_{uv}, v) connecting the corresponding intermediate vertex to the endpoints u, v of the original edge.
- The set E_C consisting of the edges connecting the central vertices with the intermediate ones. In particular, each $i_{uv} \in V_I$ is connected with exactly one of the central vertices corresponding to the faces that (u,v) belongs to.
- The set E_H consisting of the edges connecting the existing and central vertices to the corresponding hanging ones. In particular, we connect each $w \in V_E$ (resp. $w \in V_C$) with all the hanging vertices $V_H(w)$. Then, $deg_{G'}(w)$ must be $\Gamma + 1$ (resp. $\Gamma + 2$).

The construction of G' is completed by removing any central vertices (and the corresponding hanging ones) not connected to any intermediate vertices. Fig. 2 demonstrates a simple example of the construction above. The edges of graph G are marked with heavy lines and its vertices with bigger cycles.

By carefully studying the distance constraints in the resulting graph, we reach a radio coloring with no more than $\Gamma + 5$ colors (see [6]). Thus, we have:

Claim. If the graph G is 3-colorable, then the suggested radiocoloring for the resulting graph G' has span no more than $\Gamma + 5$. (See [6] for a proof.) □

Claim. If there exists a radiocoloring of G' of span $\Gamma + 5$ then G is 3-colorable. (See [6] for a proof.) □

(end of proof of Theorem 1) □

Corollary 1. *The min span order radiocoloring problem is NP-complete for planar graphs.* □

3 A Constant Ratio Approximation Algorithm

We provide here an approximation algorithm for radiocoloring of planar graphs by modifying the constructive proof of the theorem presented by Heuvel and McGuiness in [8]. Our algorithm is easier to verify with respect to correctness than what the proof given by [8] suggests. It also has better time complexity (i.e. $O(n\Delta)$) compared to the (implicit) algorithm in [8] which needs $O(n^2)$ and also assumes that a planar embedding of the graph is given. The improvement was achieved by performing the heavy part of the computation of the algorithm only in some instances of G instead of all as in [8]. This enables less checking and computations in the algorithm. Also, the behavior of our algorithm is very simple and more time efficient for graphs of small maximum degree. Finally, the algorithm provided here needs no planar embedding of G, as opposed to the algorithm implied in [8].

Very recently and independently, Agnarsson and Halldórsson [2] presented approximations for the chromatic number of square and power graphs (G^k). Their method does not explicitly present an algorithm. A straightforward implementation is difficult and not efficient. Also, the performance ratio for planar graphs of general Δ obtained is also 2.

The theorem of [8] states that a planar graph G can be radio-colored with at most $2\Delta + 25$ colors. More specifically, [8] considers the problem of L_(p,q)-Labeling, which is defined as following:

Definition 2. L_(p,q)-Labeling *Find an assignment $\phi : V \longrightarrow \{0, 1, \ldots, \nu\}$, called L_(p,q)-Labeling, which satisfies: $|\phi(u) - \phi(v)| \geq p \;\; if \; dist_G(u, v) = 1$ and $|\phi(u) - \phi(v)| \geq q \;\; if \; dist_G(u, v) = 2$.*

Definition 3. *The minimum ν for which an L_(p,q)-Labeling exists is denoted by $\lambda(G; p, q)$ (the p,q-span of G).*

The main theorem of [8] is the following:

Theorem 2. *([8]) If G is a planar graph with maximum degree Δ and p, q are positive integers with $p \geq q$, then $\lambda(G; p, q) \leq (4\,q - 2)\,\Delta + 10\,p + 38\,q - 23$.*

By setting $p = q = 1$ and using the observation $\lambda(G; 1, 1) = \chi(G^2)$, where $\chi(G^2)$ is the usual chromatic number of the graph G^2 (defined in the Introduction section), we get immediately, as also [8] notices, that:

Corollary 2. *If G is a planar graph with maximum degree Δ then $\chi(G^2) \leq 2\Delta + 25$.*

The theorem is proved using two Lemmata. The first of the two Lemmata is the following:

Lemma 1. *([8]) Let G be a simple planar graph. Then there exists a vertex v with k neighbors $v_1, v_2, \ldots, v_k$ with $d(v_1) \leq \cdots \leq d(v_k)$ such that one of the following is true:*
(i) $k \leq 2$; (ii) $k = 3$ with $d(v_1) \leq 11$; (iii) $k = 4$ with $d(v_1) \leq 7$ and $d(v_2) \leq 11$; (iv) $k = 5$ with $d(v_1) \leq 6$, $d(v_2) \leq 7$, and $d(v_3) \leq 11$.

The second Lemma, which is quite similar (see [8]). These two Lemmata give the so-called *unavoidable configurations* of G. The following operations apply to G: For an edge $e \in E$ let G/e denote the graph obtained from G by contracting e. For a vertex $v \in V$ let $G * v$ denote the graph obtained by deleting v and for each $u \in N(v)$ adding an edge between u and u^- and between u and u^+ (if these edges do not exist in G already). The notation $N(v)$ denotes the neighbors of v. The notation u^-, with $u^- \in N(v)$, denotes the edge $vu-$ which directly precedes edge vu (moving clockwise), and u^+, with $u^+ \in N(v)$, refers to the edge $vu+$ which directly succeeds edge vu.

The two Lemmata are used to define the graph H, a vertex $v \in V(G)$ and an edge $e \in E(G)$ using the rules explained in [8]. The main idea is to define H to be $H = G/e$ or $H = G * v$, with $e = vv_1$ and $d(v) \leq 5$, depending on which cases of the two Lemmata holds, so that always $\Delta(H) \leq \Delta$. Using these observations it is proved, by induction, that the minimum (p,q)-span needed for the L_(p,q)-Labeling of H is less or equal to $\lambda(G; p, q) \ \leq \ (4\,q - 2)\,\Delta + 10\,p + 38\,q - 23$.

From H we can easily return to G as follows. If $H = G/e$ then let v' the new vertex created from the contraction of edge e. In this case, in G we set $v_1 = v'$ (this is a valid assumption since $degree(v_1) \leq degree(v')$). Now we only need to color vertex v (for both cases of $H = G/e$ or $H = G * v$). From the way v was chosen, we have $d(v) \leq 5$, and it is easily seen that it can be colored with one of the $\lambda(G; p, q) \ \leq \ (4\,q - 2)\,\Delta + 10\,p + 38\,q - 23$ colors.

For the case of radiocoloring of a planar graph G, we can use p=1 and q=1 for the order, thus, the above theorem states that we need at most $2\Delta + 25$ colors.

3.1 Our Algorithm

We will use only lemma 1 and the operation G/e in order to provide a much more simple and more efficient algorithm than what implied in [8].

Radiocoloring(G)

[I] Sort the vertices of G by their degree.

[II] If $\Delta \leq 12$ then follow procedure (1) below:

Procedure (1): Every planar graph G has at least one vertex of degree ≤ 5. Now, inductively assume that any proper (in vertices) subgraph of G can be radiocolored by 66 colors. Consider a vertex v in G with $degree(v) \leq 5$. Delete v from G to get G'. Now recursively radiocolor G' with 66 colors. The number of colors that v has to avoid is at most $5\Delta + 5 \leq 65$. Thus, there is one free color for v.

[III] If $\Delta > 12$ then

1. Find a vertex v and a neighbor v_1 of it, as described in Lemma 1, and set $e = vv_1$.
2. Form $G' = G/e$ ($G' = (V', E')$ with $|V'| = n - 1$, while $|V| = n$) and denote the new vertex in G' obtained by the contraction of edge e as v'.
   Modify the sorted degrees of G by deleting v, v_1, and inserting v' at the appropriate place, and also modify the possible affected degrees of the neighbors of both v and v_1.

```
3. φ(G') =Radiocoloring(G')
4. Extend φ(G') to a valid radiocoloring of G:
(a) Set v_1 = v' and give to v_1 the color of v'.
(b) Color v with one of the colors used in the radiocoloring φ of G'.
```

3.2 Correctness, Time Efficiency and Approximation Ratio of the Algorithm

Notice first for procedure (1) that it implies a proper coloring of G^2 with $X = 66$ colors. Then, assign the frequencies $1, 3, \ldots, 2X-1$ to the obtained color classes of G^2. This is a proper radiocoloring of G with the same number of colors.

Proposition 1. *The algorithm radiocoloring(G) outputs a valid radiocoloring for G using no more than $max\{66, 2\Delta + 25\}$ colors.*

Proof. By induction, see [6] for details. □

Lemma 2. *Our algorithm approximates $X_{order}(G)$ by a constant factor of at most $max\{2 + \frac{25}{\Delta}, \frac{66}{\Delta}\}$.* □

Lemma 3. *Our algorithm runs in $O(n\Delta)$ sequential time.*
(See [6] for a proof.) □

4 An fpras for the Number of Radiocolorings of a Planar Graph

4.1 Sampling and Counting

Let G be a *planar* graph of maximum degree $\Delta = \Delta(G)$ on vertex set $V = \{0, 1, \ldots, n-1\}$ and C be a set of λ colors. Let $\phi : V \to C$ be a (proper) radiocoloring of the vertices of G. Such a radiocoloring always exists if $\lambda \geq 2\Delta + 25$ and can be found by our $O(n\Delta)$ time algorithm of the previous section.

Consider the Markov Chain (X_t) whose state space $\Omega = \Omega_\lambda(G)$ is the set of all radiocolorings of G with λ colors and whose transition probabilities from radiocoloring X_t are modelled by:

1. choose a vertex $v \in V$ and a color $c \in C$ uniformly at random (u.a.r.)
2. recolor vertex v with color c. If the resulting coloring X' is a *proper* radiocoloring then let $X_{t+1} = X'$ else $X_{t+1} = X_t$.

The procedure above is similar to the "Glauber Dynamics" of an antiferromagnetic Potts model at zero temperature, and was used by [9] to estimate the number of proper colorings of any low degree graph with k colors.

The Markov Chain (X_t), which we refer to in the sequel as $M(G, \lambda)$, is *ergodic*, provided $\lambda \geq 2\Delta + 26$, in which case its stationary distribution is *uniform* over Ω. We show here that $M(G, \lambda)$ is *rapidly mixing* i.e. converges, in time polynomial in n, to a close approximation of the stationary distribution, provided that $\lambda \geq 2(2\Delta + 25)$. This can be used to get a fully polynomial randomised approximation scheme (fpras) for the number of radiocolorings of a planar graph G with λ colors, in the case where $\lambda \geq 4\Delta + 50$. For some definitions and measures used below see [6].

4.2 Rapid Mixing

As indicated by the (by now standard) techniques for showing rapid mixing by *coupling* ([9, 10]), our strategy here is to construct a coupling for $M = M(G, \lambda)$, i.e. a stochastic process (X_t, Y_t) on $\Omega \times \Omega$ such that each of the processes (X_t), (Y_t), considered in isolation, is a faithful copy of M. We will arrange a joint probability space for (X_t), (Y_t) so that, far from being independent, the two processes tend to *couple* so that $X_t = Y_t$ for t large enough. If coupling can occur rapidly (independently of the initial states X_0, Y_0), we can infer that M is rapidly mixing, because the variation distance of M from the stationary distribution is bounded above by the probability that (X_t) and (Y_t) *have not* coupled by time t (see e.g. [1, 4]).

The transition $(X_t, Y_t) \to (X_{t+1}, Y_{t+1})$ in the coupling is defined by the following experiment:

1. Select $v \in V$ uniformly at random (u.a.r.).

2. Compute a permutation $g(G, X_t, Y_t)$ of C according to a procedure to be explained.

3. Choose a color $c \in C$ u.a.r.

4. In the radiocoloring X_t (respectively Y_t) recolor vertex v with color c (respectively $g(c)$) to get a new radiocoloring X' (respectively Y')

5. If X' (respectively Y') is a proper radiocoloring then $X_{t+1} = X'$ (respectively $Y_{t+1} = Y'$), else let $X_{t+1} = X_t$ (respectively $Y_{t+1} = Y_t$).

Note that, whatever procedure is used to select the permutation g, the distribution of $g(c)$ is *uniform*, thus (X_t) and (Y_t) are both faithful copies of M.

We now remark that any set of vertices $F \subseteq V$ can have the same color in the graph G^2 only if they can have the same color in some radiocoloring of G. Thus, given a proper coloring of G^2 with λ' colors, we can construct a proper radiocoloring of G by giving the values (new colors) $1, 3, \ldots, 2\lambda' - 1$ in the color classes of G^2. Note that this transformation preserves the number of colors (but not the span).

Now let $A = A_t \subseteq V$ be the set of vertices on which the colorings of G^2 implied by X_t, Y_t agree and $D = D_t \subseteq V$ be the set on which they disagree. Let $d'(v)$ be the number of edges incident at v in G^2 that have one point in A and one in D. Clearly, if m' is the number of edges of G^2 spanning A, D, we get $\sum_{v \in A} d'(v) = \sum_{v \in D} d'(v) = m'$.

The procedure to compute $g(G, X_t, Y_t)$ is as follows:

(a) If $v \in D$ then g is the identity.

(b) If $v \in A$ then proceed as follows: Denote by N the set of neighbors of v in G^2. Define $C_x \subseteq C$ to be the set of all colors c, such that some vertex in N receives c in radiocoloring Y_t but no vertex in N receives c in radiocoloring Y_t. Let C_y be defined as C_x with the roles of X_t, Y_t interchanged. Observe $C_x \cap C_y = \emptyset$ and $|C_x|, |C_y| \leq d'(v)$. Let, w.l.o.g., $|C_x| \leq |C_y|$. Choose any subset $C'_y \subseteq C_y$ with $|C'_y| \leq |C_x|$ and let $C_x = \{c_1, \ldots, c_r\}, C'_y = \{c'_1, \ldots, c'_r\}$ be enumerations of $C_x, C_{y'}$ coming from the orderings of X_t, Y_t. Finally, let g be the permutation $(c_1, c'_1), \ldots, (c_r, c'_r)$ which interchanges the color sets $C_x, C_{y'}$ and leaves all other colors fixed.

It is clear that $|D_{t+1}|-|D_t| \in \{-1,0,1\}$. By careful and nontrivial estimations of the probability of $\Pr\{|D_{t+1}| = |D_t| \pm 1\}$ we finally get (see [6] for a full proof): $\Pr\{D_t \neq 0\} \leq n(1-\alpha)^t \leq ne^{-\alpha t}$, where $\alpha = [\lambda - 2(2\Delta + 25)]/n > 0$ when $\lambda > 2(2\Delta + 25)$.

So, we note that $\Pr\{D_t \neq \emptyset\} \leq \epsilon$, where ϵ is upper bound on the variation distance at time t, provided that $t \geq \frac{1}{\alpha} \ln\left(\frac{n}{\epsilon}\right)$.

Theorem 3. *The above method leads to a fully polynomial randomized approximation scheme for the number of radiocolorings of a planar graph G with λ colors, provided that $\lambda > 2(2\Delta + 25)$, where Δ is the maximum degree of G.*
(See [6] for a proof.) □

5 Further Work

A major open problem is to get a polynomial time approximation to $X_{order}(G)$ of asymptotic ratio < 2. The improvement of the time efficiency of the approximation procedure is also a subject of further work.

References

1. D. Aldous: Random walks in finite groups and rapidly mixing Markov Chains. Seminaire de Probabilites XVII 1981/82 (A. Dold and B. Eckmann, eds), Springer Lecture Notes in Mathematis 986 (1982) 243-297 .
2. Geir Agnarsson, Magnus M.Hallorsson: Coloring Powers of planar graphs. Symposium of Discrete Algorithms (2000).
3. Alan A. Bertossi and Maurizio A. Bonuccelli: Code assignment for hidden terminal interference avoidance in multihop packet radio networks. IEEE/ACM Trans. Networking 3, 4 (Aug. 1995) 441 - 449.
4. P. Diaconis: Group representations in probability and statistics. Institute of Mathematical Statistics, Hayward CA, (1988).
5. D. Fotakis, G. Pantziou, G. Pentaris and P. Spirakis: Frequency Assignment in Mobile and Radio Networks. Networks in Distributed Computing, DIMACS Series in Discrete Mathematics and Theoretical Computer Science 45 American Mathematical Society (1999) 73-90.
6. D.A. Fotakis, S.E. Nikoletseas, V.G. Papadopoulou and P.G. Spirakis: NP-completeness Results and Efficient Approximations for Radiocoloring in Planar Graphs. CTI Technical Report (2000) (url: http://www.cti.gr/RD1).
7. D. Fotakis and P. Spirakis: Assignment of Reusable and Non-Reusable Frequencies. International Conference on Combinatorial and Global Optimization (1998).
8. J. Van D. Heuvel and S. McGuiness: Colouring the square of a Planar Graph. CDAM Research Report Series, Jule (1999).
9. M. Jerrum: A very simple algorithm for estimating the number of k-colourings of a low degree graph. Random Structures and Algorithms 7 (1994) 157-165.
10. M. Jerrum: Markov Chain Monte Carlo Method. Probabilistic Methods for Algorithmic Discrete Mathematics, Springer (1998).
11. S. Ramanathan, E. R. Loyd: Scheduling algorithms for Multi-hop Radio Networks. IEEE/ACM Trans. on Networking, 1(2): (April 1993) 166-172.
12. S. Ramanathan, E. R. Loyd: The complexity of distance2-coloring. 4th International Conference of Computing and information, (1992) 71-74.