A Hamiltonian Approach to the Assignment of Non-Reusable Frequencies*

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Abstract. The problem of Radio Labelling is to assign distinct integer labels to all the vertices of a graph, such that adjacent vertices get labels at distance at least two. The objective is to minimize the label span. Radio labelling is a combinatorial model for frequency assignment in case that the transmitters are not allowed to operate at the same channel.

We show that radio labelling is related to TSP(1,2). Hence, it is $\mathcal{NP}$-complete and MAX–SNP-hard. Then, we present a polynomial-time algorithm for computing an optimal radio labelling, given a coloring of the graph with constant number of colors. Thus, we prove that radio labelling is in $\mathcal{P}$ for planar graphs. We also obtain a $\frac{3}{2}$-approximation $\mathcal{NC}$ algorithm and we prove that approximating radio labelling in graphs of bounded maximum degree is essentially as hard as in general graphs.

We obtain similar results for TSP(1,2). In particular, we present the first $\frac{3}{2}$-approximation $\mathcal{NC}$ algorithm for TSP(1,2), and we prove that dense instances of TSP(1,2) do not admit a PTAS, unless $\mathcal{P} = \mathcal{NP}$.

1 Introduction

The Frequency Assignment Problem (FAP) arises from the fact that radio transmitters operating at the same or closely related frequency channels have the potential to interfere with each other. FAP can be formulated as an optimization problem as follows: Given a collection of transmitters to be assigned operating channels and a set of interference constraints on transmitter pairs, find an assignment that fulfills all the interference constraints and minimizes the allocated bandwidth.

A common model for FAP is the *interference graph*. Each vertex of an interference graph represents a transmitter, while each edge represents an interference constraint between the adjacent transmitters. The frequency channels are usually assumed to be uniformly spaced in the spectrum and are labelled using positive integers. Frequency channels with adjacent integer labels are assumed adjacent in the spectrum [5, 6, 8]. Clearly, FAP is a generalization of graph coloring [6].

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Instead of specifying the interference constraints for each pair of transmitters, FAP can be defined by specifying a minimum allowed spatial distance for each channel/spectral separation that is a potential source of interference [5, 6]. In [6] these are called *Frequency-Distance* (FD) constraints. FD-constraints can be given as a set of distances $\{D_0, D_1, \ldots, D_\kappa\}$, $D_0 \geq D_1 \geq \cdots \geq D_\kappa$, where the distance D_x, $x \in \{0, 1, \ldots, \kappa\}$, is the minimum distance between transmitters using channels at distance x.

The FD(κ)-coloring problem has been proposed as a model for FAP in unweighted interference graphs [8]. In FD(κ)-coloring we seek a function $\chi_\kappa : V \mapsto \{1, \ldots, \nu\}$ that fulfills the FD-constraints with respect to the graph distances $D_0 = \kappa + 1, D_1 = \kappa, \ldots, D_\kappa = 1$, and minimizes the largest color ν used (color span). Alternatively, $v, u \in V$ are only allowed to get colors at distance x, $|\chi_\kappa(v) - \chi_\kappa(u)| = x$, $x \in \{0, 1, \ldots, \kappa\}$, if v and u are at distance at least $\kappa - x + 1$ from each other in the interference graph. A polynomial-time exact algorithm for a variant of FD(κ)-coloring in lattices is presented in [8]. FD(2) -coloring ($\kappa = 2$) is a combinatorial model for the widely used "co-channel" and "adjacent-channel" interference constraints. FD(2)-coloring is also called *radio coloring* in [4]. A problem similar to FD(2)-coloring is studied [11] in the context of mobile networks, where each vertex may require more than one colors (multi-coloring). A polynomial-time approximation algorithm for triangular lattices is presented in [11].

In some practical applications the transmitters cover a local or metropolitan area. Hence, the transmitters are not allowed to operate at the same channel ($D_0 = \infty$, non-reusable frequency channels). In this paper we study *Radio Labelling*, which is the equivalent of FD(2)-coloring in the context of non-reusable frequency assignment. In particular, a valid radio labelling fulfills the FD-constraints with respect to $D_0 = \infty, D_1 = 2, D_2 = 1$. In radio labelling we seek an assignment of distinct integer labels to all the vertices of a graph, such that adjacent vertices get labels at distance at least two. The objective is to minimize the maximum label used (label span). The definition of radio labelling is communicated to us by [7].

Competitive algorithms and lower bounds for on-line radio labelling are presented in [4]. Moreover, radio labelling and similar combinatorial models for non-reusable frequency assignment are used for obtaining lower bounds on the optimal values of general FAP instances [1, 14]. These lower bounds are necessary for assessing the quality of the solutions found by heuristic algorithms.

1.1 Summary of Results

We start with proving that radio labelling is equivalent to Hamiltonian Path with distances one and two (HP(1,2)). Hence, radio labelling is MAX–SNP-hard and approximable in polynomial time within $\frac{7}{6}$ [13]. We also prove that both radio labelling and FD(2)-coloring remain $\mathcal{NP}$-complete for graphs of diameter two.

Next, we present a polynomial-time algorithm for computing an optimal radio labelling, given a coloring of the graph with constant number of colors. Therefore, we prove that radio labelling is in $\mathcal{P}$ for planar graphs and graphs colorable with constant number of colors in polynomial time.

We proceed to deal with radio labelling in general graphs and we obtain a $\frac{3}{2}$-approximation $\mathcal{NC}$ algorithm. In particular, if MC_G denotes the cardinality of the maximum clique and $\chi(G)$ denotes the chromatic number of the graph $G(V,E)$, the algorithm produces a radio labelling of value at most $|V| + \mathrm{MC}_G - 1 \leq |V| + \chi(G) - 1$, without assuming any knowledge on MC_G or $\chi(G)$. Therefore, it outperforms all known sequential approximation algorithms for graphs with clique number (chromatic number) less than $\frac{|V|}{6}$.

Additionally, we show that there exists a constant $\Delta^\star < 1$, such that, for all $\Delta \in [\Delta^\star, 1)$, radio labelling in graphs $G(V,E)$ of maximum degree $\Delta(G) \leq \Delta|V|$ is essentially as hard to approximate as in general graphs. This is in contrast with the fact that, if $\Delta(G) < \frac{n}{2}$, then an optimal radio labelling of value $|V|$ can be easily computed.

Since radio labelling is closely related to HP(1,2) and TSP(1,2), we also obtain similar results for these problems. In particular, given a partition of a graph G into constant number of cliques, we show how to decide in polynomial time if G is Hamiltonian. Moreover, we present the first $\frac{3}{2}$-approximation $\mathcal{NC}$ algorithm for TSP(1,2) and HP(1,2). The best previously known parallel algorithm for metric TSP and all the special cases is the algorithm of Christofides [2], which is in $\mathcal{RNC}$, but it is not known to be in $\mathcal{NC}$.

We also prove that dense instances of TSP(1,2) and HP(1,2) are essentially as hard to approximate as general instances, as opposed to the success in designing Polynomial Time Approximation Schemes (PTAS) for dense instances of many combinatorial optimization problems (cf. [10] for a survey). Recently a different proof of the same result appears independently in [3].

2 Preliminaries

Given a graph $G(V,E)$, $\chi(G)$ denotes the chromatic number, MIS_G is the cardinality of the Maximum Independent Set, MC_G is the cardinality of the Maximum Clique, and $d(v,u)$ denotes the length of the shortest path between $v, u \in V$. For all graphs G, $\mathrm{MC}_G \leq \chi(G)$. The complementary graph $\overline{G}$ is a graph on the same vertex set V that contains an edge $\{v,u\}$, iff $\{v,u\} \notin E$. Clearly, $\mathrm{MIS}_G = \mathrm{MC}_{\overline{G}}$. A graph $G(V,E)$ is called *Δ-bounded*, for some constant $1 > \Delta > 0$, if the maximum degree $\Delta(G)$ is no more than $\Delta|V|$. Also, a graph $G(V,E)$ is called *δ-dense*, for some constant $1 > \delta > 0$, if the minimum degree $\delta(G)$ is at least $\delta|V|$.

The Travelling Salesman Problem with distances one and two (TSP(1,2)) is the problem of finding a Hamiltonian cycle of minimum length in a complete graph, where all the edge lengths are either one or two. Clearly, given a graph $G(V,E)$, TSP(1,2) is a generalization of the Hamiltonian cycle problem in G, where each edge in E is considered of length one and each edge not in E (non-edge) of length two. TSP_G denotes the optimal value of the TSP(1,2) instance corresponding to the graph G. An interesting variant of TSP(1,2) is Hamiltonian Path with distances one and two (HP(1,2)), where the objective is to find a Hamiltonian path of minimum length. Since TSP(1,2) and HP(1,2) generalize the Hamiltonian cycle and path problems respectively, they are $\mathcal{NP}$-complete. Moreover, they are MAX–SNP-hard and there exists a polynomial-time $\frac{7}{6}$-approximation algorithm [13].

Definition 1. (FD(κ)-labelling)
INSTANCE: A graph $G(V, E)$.
SOLUTION: A valid FD(κ)-labelling, i.e. a function $\mathrm{L}_\kappa : V \mapsto \{1, \ldots, \nu\}$ such that, for all $v, u \in V$, $|\mathrm{L}_\kappa(v) - \mathrm{L}_\kappa(u)| = x$, $x \in \{1, \ldots, \kappa\}$, only if $d(v, u) \geq \kappa - x + 1$.
OBJECTIVE: Minimize the maximum label ν used (label span).

This paper is devoted to FD(2)-labelling, which is also called *Radio Labelling*. In the following, given a graph $G(V, E)$, $\mathrm{RL}(G)$ denotes the value of an optimal radio labelling for G and, for any $v \in V$, $\mathrm{RL}(v)$ denotes the label assigned to v by a valid radio labelling.

Since radio labelling assigns distinct integer labels to all the vertices of a graph, it is a vertex arrangement problem. In particular, given an arrangement $v_1, v_2, \ldots, v_n$, a valid radio labelling can be computed as follows: $\mathrm{RL}(v_1) = 1$, and for $i = 1, \ldots, n-1$, $\mathrm{RL}(v_{i+1}) = \mathrm{RL}(v_i) + 1$, if $\{v_i, v_{i+1}\} \notin E$, and $\mathrm{RL}(v_{i+1}) = \mathrm{RL}(v_i) + 2$, otherwise. Conversely, any radio labelling implies a vertex arrangement, in the sense that v precedes u, iff $\mathrm{RL}(v) < \mathrm{RL}(u)$. A radio labelling L is minimal, if there does not exist another radio labelling L' that corresponds to the same vertex arrangement and $\mathrm{RL}(L') < \mathrm{RL}(L)$. In the sequel, we only consider minimal radio labellings. It is not hard to verify the following.

Lemma 2. *Radio labelling is equivalent to HP(1,2) in the complementary graph.*

Hence, Dirac's Theorem implies that if $\Delta(G) < \frac{|V|}{2}$, a radio labelling of value $|V|$ exist and can be found in polynomial time. Given a coloring of G with χ colors, it is easy to find a radio labelling of value at most $|V| + \chi - 1$. Therefore, $|V| \leq \mathrm{RL}(G) \leq |V| + \chi(G) - 1$. Additionally, for graphs of diameter two, radio labelling is equivalent to FD(2)-coloring.

Lemma 3. *Radio labelling and* FD(2)*-coloring restricted to graphs of diameter two are $\mathcal{NP}$-complete.*

Proof sketch. Given a graph G of $\mathrm{diam}(G) = 2$, the problem of deciding if the complementary graph $\overline{G}$ contains a Hamiltonian path is $\mathcal{NP}$-complete. □

3 An Exact Algorithm for Constant Number of Colors

Since radio labelling is equivalent to HP(1,2) and a coloring corresponds to a partition into cliques of the complementary graph, we present the technical part of the proof in the context of Hamiltonian paths/cycles and partitions into cliques. In particular, we prove that, given a partition of a graph $G(V, E)$ into constant number of cliques, we can decide if G is Hamiltonian in polynomial time.

3.1 Hamiltonian Cycles and Partitions into Cliques

Theorem 4. *Given a graph $G(V,E)$, $|V| = n$, and a partition of V into $\kappa > 1$ cliques, there exists a deterministic algorithm that runs in time $\mathcal{O}\big(n^{\kappa(2\kappa-1)}\big)$ and decides if G is Hamiltonian. If G is Hamiltonian, the algorithm outputs a Hamiltonian cycle.*

Proof. A set of inter-clique edges $M \subseteq E$ is an HC-set if M can be extended to a Hamiltonian cycle using only clique-edges, i.e. there exists a $M^{(c)} \subseteq E$ of clique edges such that $M \cup M^{(c)}$ is a Hamiltonian cycle.

We first show that, given a set of inter-clique edges M, we can decide if M is an HC-set and construct a Hamiltonian cycle from M in poly(n,κ) time (Proposition 5). Then, we prove that G is Hamiltonian iff there exists an HC-set of cardinality at most $\kappa(\kappa-1)$ (Lemmas 6 and 7). The algorithm exhaustively searches all the sets of inter-clique edges of cardinality at most $\kappa(\kappa-1)$ for an HC-set. Additionally, we conjecture that, if G is Hamiltonian, then there exists an HC-set of cardinality at most $2(\kappa-1)$. We prove this conjecture for a special case (Lemma 8) and we use Lemma 6 to show the equivalence to Conjecture 9.

Let $\mathcal{C} = \{C_1, \ldots, C_\kappa\}$ be a partition of V into $\kappa > 1$ cliques. Given a set $M \subseteq E$ of inter-clique edges, the *clique graph* $T(C,M)$ contains exactly κ vertices, that correspond to the cliques of $\mathcal{C}$, and represents how the edges of M connect the different cliques. If M is an HC-set, then the corresponding clique graph $T(C,M)$ is connected and eulerian. However, the converse is not always true.

Given a set of inter-clique edges M, we color an edge RED, if it shares a vertex of G with another edge of M. Otherwise, we color it BLUE. The corresponding edges of G are colored with the same colors, while the remaining edges $(E-M)$ are colored BLACK. Additionally, we color RED each vertex $v \in V$, which is the common end vertex of two or more RED edges. We color BLUE each vertex $v \in V$ to which exactly one edge of M (RED or BLUE) is incident. The remaining vertices of G are colored BLACK (Figure 1).

Let H be any Hamiltonian cycle of G and let M be the corresponding set of inter-clique edges. Obviously, RED vertices cannot be exploited for visiting any BLACK vertices belonging to the same clique. If H visits a clique C_i through a vertex v, and leaves C_i through a vertex u, then $v, u \in C_i$ consist a BLUE vertex pair. A BLUE pass through a clique C_i is a simple path of length at least one, that entirely consists of non-RED vertices of C_i. A clique C_i is covered by M, if all the vertices of C_i have degree at most two in M, and the existence of a non-RED vertex implies the existence of at least one BLUE vertex pair. The following proposition characterizes HC-sets.

Proposition 5. *A set of inter-clique edges M is an HC-set iff the corresponding clique graph $T(C,M)$ is connected, eulerian, and,*

(a) For all $i = 1, \ldots, \kappa$, C_i is covered by M; and
(b) There exists an eulerian trail R for T such that: For any RED *vertex $v \in V$, R passes through v exactly once using the corresponding* RED *edge pair.*

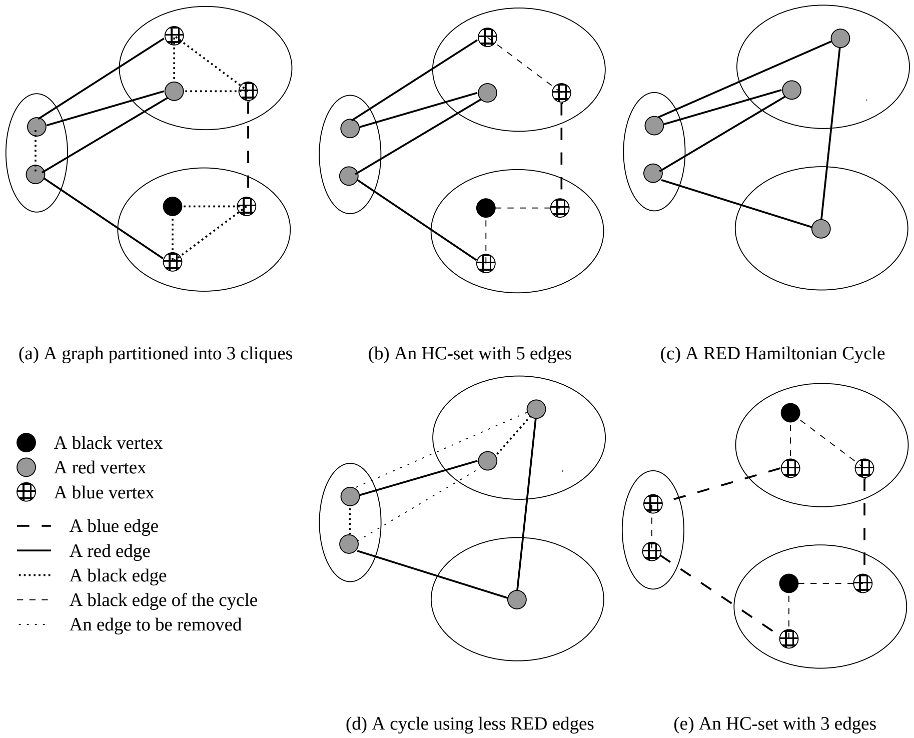

Fig. 1. An application of Lemmas 6 and 7.

Proof sketch: Any HC-set corresponds to a connected, eulerian clique graph $T(C, M)$ that fulfills both (a) and (b). Conversely, we can extend M into a Hamiltonian cycle H following the eulerian trail R. All the RED vertices can be included in H with degree two because of (b). Moreover, since R is an eulerian trail and (a) holds for M, all the BLUE and the BLACK vertices can be included in H with degree two. Therefore, H is a Hamiltonian cycle. $\square$

The proof of Proposition 5 implies a deterministic procedure for deciding if a set of inter-clique edges is an HC-set in $\mathrm{poly}(n, \kappa)$ time. Moreover, in case that M is an HC-set, this procedure outputs a Hamiltonian cycle.

Lemma 6. *Let $B_\kappa \geq 2$ be some integer only depending on κ such that, for any graph $G(V, E)$ and any partition of V into κ cliques, if G is Hamiltonian and $|V| > B_\kappa$, then G contains at least one Hamiltonian cycle not entirely consisting of inter-clique edges (RED vertices).*
Then, for any graph $G(V, E)$ and any partition of V into κ cliques, G is Hamiltonian iff it contains a Hamiltonian cycle with at most B_κ inter-clique edges.

Proof. Let H be the Hamiltonian cycle of G containing the minimum number of inter-clique edges and let M be the corresponding set of inter-clique edges.

Assume that $|M| > B_\kappa$. The hypothesis implies that H cannot entirely consist of RED vertices. Therefore, H should contain at least one BLUE vertex pair.

We substitute any BLUE pass of H through a clique C_i with a single RED super-vertex $\hat{v}$, that also belongs to the clique C_i. Hence, $\hat{v}$ is connected to all the remaining vertices of C_i using BLACK edges. These substitutions result in a cycle H' that entirely consists of RED vertices, and contains exactly the same set M of inter-clique edges with H.

Obviously, the substitutions of all the BLUE passes of H with RED super-vertices result in a graph $G'(V', E')$ that is also Hamiltonian, $|V'| > B_\kappa$, and V' is partitioned into κ cliques. Moreover, for any Hamiltonian cycle $H_{G'}$ of G', the reverse substitutions of all the RED super-vertices $\hat{v}$ with the corresponding BLUE passes result in a Hamiltonian cycle of G that contains exactly the same set of inter-clique edges with $H_{G'}$ (Figure 1).

Since H' is a Hamiltonian cycle that entirely consists of inter-clique edges and $|V'| > B_\kappa$, the hypothesis implies that there exists another Hamiltonian cycle of G' that contains strictly less inter-clique edges than H'. Therefore, there exists a Hamiltonian cycle of G that contains less inter-clique edges than H. □

Lemma 6 implies that, in order to prove the upper bound on the cardinality of a minimum HC-set, it suffices to prove the same upper bound on the number of vertices of Hamiltonian graphs that (i) can be partitioned into κ cliques, and (ii) only contain Hamiltonian cycles entirely consisting of inter-clique edges. It should be intuitively clear that such graphs cannot contain an arbitrarily large number of vertices.

Lemma 7. *Given a graph $G(V, E)$ and a partition of V into κ cliques, G is Hamiltonian iff there exists an HC-set M such that $|M| \leq \kappa(\kappa - 1)$.*

Proof. By definition, the existence of an HC-set implies that G is Hamiltonian. Conversely, let H be the Hamiltonian cycle of G that contains the minimum number of inter-clique edges, and let M be the corresponding HC-set. If $|M| \leq \kappa(\kappa-1)$, then we are done. Otherwise, Lemma 6 implies that it suffices to prove the same upper bound on $|V|$ for graphs $G(V, E)$ that only contain Hamiltonian cycles entirely consisting of inter-clique edges.

Assume that $|V| = |M|$ and the coloring of V under M entirely consists of RED vertices, and consider an arbitrary orientation of the Hamiltonian cycle H (e.g. a traversal of the edges of H in the clockwise direction). If there exist a pair of cliques C_i and C_j and four vertices $v_1, v_2 \in C_i$ and $u_1, u_2 \in C_j$, such that both v_x are followed by u_x ($x = 1, 2$) in a traversal of H, then the BLACK edges $\{v_1, v_2\}$ and $\{u_1, u_2\}$ can be used instead of $\{v_1, u_1\}$ and $\{v_2, u_2\}$ in order to obtain a Hamiltonian cycle containing less inter-clique edges than H. The previous situation can be avoided only if, for all $i = 1, \ldots, \kappa$, and $j = 1, \ldots, \kappa$, $j \neq i$, at most one vertex $v_j \in C_i$ is followed by a vertex $u \in C_j$ in any traversal of H. Hence, if $|V| > \kappa(\kappa-1)$, then G contains at least one Hamiltonian cycle not entirely consisting of inter-clique edges. Alternatively, any HC-set M of minimum cardinality contains at most two edges between any pair of cliques C_i and C_j. Thus, M contains at most $\kappa(\kappa - 1)$ inter-clique edges. □

Therefore, we can decide if G is Hamiltonian in time $\mathcal{O}\big(n^{\kappa(2\kappa-1)}\big)$, because the number of the different edge sets containing at most $\kappa(\kappa-1)$ inter-clique edges is at most $n^{2\kappa(\kappa-1)}$, and we can decide if a set of inter-clique edges is an HC-set in time $\text{poly}(n,\kappa) = \mathcal{O}(n^{\kappa})$. □

A BLUE Hamiltonian cycle is a Hamiltonian cycle that does not contain any RED vertices or edges. We can substantially improve the bound of $\kappa(\kappa-1)$ for graphs containing a BLUE Hamiltonian cycle.

Lemma 8. *Given a graph $G(V,E)$ and a partition of V into κ cliques, if G contains a* BLUE *Hamiltonian cycle, then there exists an HC-set M entirely consisting of* BLUE *edges, such that $|M| \le 2(\kappa-1)$.*

Proof. Let H be the BLUE Hamiltonian cycle of G that contains the minimum number of inter-clique edges and let M be the corresponding HC-set. Assume that $|M| > 2(\kappa-1)$, otherwise we are done. Notice that, since RED vertices cannot be created by removing edges, any $M' \subseteq M$ that corresponds to an eulerian, connected, clique graph $T(C,M')$ is an HC-set that only contains BLUE edges.

Let $S_T(C,M_S)$ be any spanning tree of $T(C,M)$. Since $|M_S| = \kappa-1$, the graph $T^{(S)}(C, M-M_S)$, which is obtained by removing the edges of the spanning tree from T, contains at least κ edges. Therefore, $T^{(S)}$ contains a simple cycle L. The removal of the edges of L does not affect connectivity (the edges of L do not touch the spanning tree S_T), and subtracts two from the degrees of the involved vertices/cliques. Clearly, the clique graph $T'(C, M-L)$ is connected and eulerian, and $M-L$, $|M-L| < |M|$, is an HC-set. □

There exist HC-sets of cardinality exactly $2(\kappa-1)$ that correspond to a Hamiltonian cycle using the minimum number of inter-clique edges. Therefore, the bound of $2(\kappa-1)$ is tight. However, we are not able to construct HC-sets that contain more than $2(\kappa-1)$ edges and correspond to Hamiltonian cycles using the minimum number of inter-clique edges. Hence, we conjecture that the bound of $2(\kappa-1)$ holds for any graph and any partition into κ cliques. An inductive (on $|V|$) application of Lemma 6 suggests that this conjecture is equivalent to the following:

Conjecture 9. *For any Hamiltonian graph $G(V,E)$ of $2\kappa-1$ vertices and any partition of V into κ cliques, there exists at least one Hamiltonian cycle not entirely consisting of inter-clique edges.*

3.2 A Reduction from Radio Labelling to Hamiltonian Cycle

Lemma 10. *Given a graph $G(V,E)$, $|V| = n$, and a coloring of G with κ colors, an optimal radio labelling can be computed in $\mathcal{O}\big(n^{\kappa(2\kappa+1)}\big)$ time.*

Proof sketch. Since $\text{RL}(G) \le n + \kappa - 1$, optimal HP(1,2)/TSP(1,2) solutions to the complementary graph $\overline{G}$ can be found by at most $\mathcal{O}(n^{2\kappa})$ calls to the algorithm of Theorem 4. □

Since any planar graph can be colored with constant number of colors in polynomial time, the following theorem is an immediate consequence of Lemma 10.

Theorem 11. *An optimal radio labelling of a planar graph can be computed in polynomial time.*

Remark. Conjecture 9 implies that, given a graph $G(V,E)$ and a coloring with κ colors, an optimal radio labelling can be computed in time $n^{\mathcal{O}(\kappa)}$.

4 An $\mathcal{NC}$ Approximation Algorithm for General Graphs

Then, we present a $\frac{3}{2}$-approximation $\mathcal{NC}$ algorithm for radio labelling in general graphs. For clarity of presentation, the algorithm is described and analyzed in the context of Hamiltonian paths and cycles. In particular, we obtain the first $\frac{3}{2}$-approximation $\mathcal{NC}$ algorithm for HP(1,2) and TSP(1,2).

Theorem 12. *The following inequalities hold for any graph $G(V,E)$, $|V|=n$:*

$$2\mathrm{MIS}_G \leq \mathrm{TSP}_G \leq n + \mathrm{MIS}_G \tag{1}$$

Moreover, a tour and an independent set that fulfill (1) can be computed in $\mathcal{NC}$.

Proof. The algorithm TSP-MIS (Figure 2) proceeds in phases, such that the following holds for the i^{th} phase:

For all the vertices v, $v \in V^i$ implies that $\deg_P(v) \leq 1$. Moreover, for each non-trivial path $p \in P$, exactly one of the end vertices of p is in V^i.

Therefore, at the end of the algorithm, the set P consists of simple paths (of 1-edges) and isolated vertices. If the isolated vertices are considered as trivial paths, the edge set P covers V with vertex disjoint simple paths. Additionally, the vertex sets S^i are independent sets, because M^i is a maximal matching in $G^i(V^i,E^i)$.

Performance: The performance of the algorithm is determined by the number of edges (of length 1) that are included in the set P at the end of the algorithm. Initially, the set P is empty. At each phase i, the edges of M^i are added to P in step (*). Let $|M^i|$ be the number of edges of a maximal matching M^i of the graph G^i. The algorithm runs for $K+1$ phases, where K will be fixed later. Clearly, $|P| \geq \sum_{i=1}^{K} |M^i|$.

Since the vertices of V^i not covered by a maximal matching M^i form an independent set, we obtain that,

$$\frac{|V^i| - \mathrm{MIS}_G}{2} \leq |M^i| \leq \frac{|V^i|}{2}$$

Furthermore, for any pair of vertices of V^i that are matched by M^i, exactly one vertex is added to V^{i+1}. Therefore,

$$\frac{|V^i|}{2} \leq |V^{i+1}| = |V^i| - |M^i| \leq \frac{|V^i| + \mathrm{MIS}_G}{2}$$

The following inequality holds for all $i \geq 1$, and can be proved by induction.

$$|V^{i+1}| \leq \frac{|V^1| + (2^i - 1)\mathrm{MIS}_G}{2^i} \tag{2}$$

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Algorithm TSP-MIS
Input:  A graph G(V, E).
Output: A set P of edges that cover V with vertex disjoint simple paths.
        A set S* of independent vertices.
P := ∅;
i := 0;
repeat
    i := i + 1;      /* i^th phase */
    V^i := {v ∈ V : deg_P(v) = 0};
    For any non-trivial path p ∈ P,
        add exactly one of the end vertices of p to V^i;
    E^i := {{v, w} ∈ E : v ∈ V^i ∧ w ∈ V^i};
    Find a Maximal Matching M^i in G^i(V^i, E^i);
(*) P := P ∪ M^i;
    S^i := {v ∈ V^i : deg_{M^i}(v) = 0};
until M^i = ∅ or i > K;
S* := S^i of maximum cardinality;
return(P, S*);
```

Fig. 2. The Algorithm TSP-MIS.

If $K = \lceil \log n \rceil$, then inequality (2) implies that $|V^{K+1}| < \mathrm{MIS}_G + 1$. By summing the equalities $|V^{i+1}| = |V^i| - |M^i|$, for all $1 \leq i \leq K+1$, we obtain that,

$$|V^{K+1}| = |V^1| - \sum_{i=1}^{K} |M^i| \geq |V| - |P|$$

Therefore, $|P| \geq n - \mathrm{MIS}_G$. Since the set P consists of simple paths of total length at least $n - \mathrm{MIS}_G$, we can construct a Hamiltonian cycle from P by adding no more than MIS_G edges of length two. Hence, $\mathrm{TSP}_G \leq n + \mathrm{MIS}_G$.

The proof of the inequality $2\mathrm{MIS}_G \leq \mathrm{TSP}_G$ is simple. Furthermore, if the cardinality of $S^\star$ was less than $\mathrm{TSP}_G - n$, then the previous analysis implies that the result of TSP-MIS would be a Hamiltonian cycle of length less than TSP_G.

There exist graphs G such that $\mathrm{TSP}_G = n + \mathrm{MIS}_G - 1$. For example, for any integer $x > 1$, consider the graph consisting of a clique on $n - x$ vertices and an independent set of x vertices, all of them connected to a single vertex of the clique. Clearly, $\mathrm{MIS}_G = x + 1$ and $\mathrm{TSP}_G = n + x$.

Complexity: The algorithm runs for at most $\lceil \log n \rceil + 1$ phases. Since the connected components of P are simple paths, the computation of G^i can be easily implemented in $\mathcal{NC}$. The complexity of the algorithm is dominated by the computation of the maximal matching. There exist a CRCW PRAM algorithm that produces a maximal matching in $\mathcal{O}(\log^3 n)$ time using $\mathcal{O}(n^2)$ processors [9]. Moreover, the sequential complexity of TSP-MIS is $\mathcal{O}(n^2 \log n)$. □

Obviously, the arguments above also apply to HP(1,2) and to radio labelling in the complementary graph $\overline{G}$. Therefore, the following is an immediate consequence of Theorem 12:

Corollary 13. *There exists an $\mathcal{NC}$ algorithm that runs in a CRCW PRAM in time $\mathcal{O}(\log^4 n)$ using $\mathcal{O}(n^2)$ processors and approximates,*

1. *Radio labelling, HP(1,2), and TSP(1,2) within $\frac{3}{2}$.*
2. *HP(1,2) and TSP(1,2) restricted to graphs G, such that $\mathrm{MIS}_G \leq \alpha n$, within $(1+\alpha)$.*
3. *Radio labelling restricted to graphs G, such that $\mathrm{MC}_G \leq \alpha n$ ($\chi(G) \leq \alpha n$), within $(1+\alpha)$.*
4. *Maximum Clique restricted to graphs G, such that $\mathrm{RL}(G) \geq (1+\beta)n$, within $\frac{2\beta}{1+\beta}$.*

5 Graphs of Bounded Maximum Degree

Even though radio labelling is in $\mathcal{P}$ for graphs $G(V,E)$ of $\Delta(G) < \frac{|V|}{2}$, we prove that there exists a constant $\frac{1}{2} < \Delta^\star < 1$, such that, for all $\Delta \in [\Delta^\star, 1)$, Δ-bounded instances of radio labelling (i.e. $\Delta(G) \leq \Delta|V|$) do not admit a PTAS.

Theorem 14. *There exists a constant $\frac{1}{2} < \Delta^\star < 1$, such that, for all $\Delta \in [\Delta^\star, 1)$, Δ-bounded instances of radio labelling do not admit a PTAS, unless $\mathcal{P} = \mathcal{NP}$.*

Proof sketch. Transform any graph $G(V,E)$ to a Δ-bounded graph by adding a clique on $\frac{1-\Delta}{\Delta}$ vertices. Since radio labelling is MAX–SNP-hard, there exists a constant $\epsilon^\star > 0$, such that radio labelling is not approximable within $(1+\epsilon^\star)$, unless $\mathcal{P} = \mathcal{NP}$. For any small constant α, $\epsilon^\star > \alpha > 0$, $\Delta^\star = \frac{1+\alpha}{1+\epsilon^\star} < 1$. □

For graphs with optimal values $\mathrm{RL}(G) \geq \frac{|V|}{\Delta}$, $\Delta \in [\Delta^\star, 1)$, the previous transformation is an L-reduction from such instances of radio labelling to Δ-bounded instances. A similar result also holds for dense instances of HP(1,2) and TSP(1,2).

Corollary 15. *There exists a constant $\delta^\star > 0$, such that, for all $0 < \delta \leq \delta^\star$, δ-dense instances of HP(1,2) and TSP(1,2) do not admit a PTAS, unless $\mathcal{P} = \mathcal{NP}$.*

6 Open Problems

An interesting direction for further research is to obtain a polynomial-time approximation algorithm for FD(2)-coloring in planar graphs using our exact algorithm for radio labelling. One approach may be to decompose the planar graph to subgraphs, such that almost all vertices of each subgraph get distinct colors by an optimal (or near optimal) assignment. Then, our exact algorithm can be used for computing an optimal radio labelling for each of the resulting subgraphs. Obviously, the decomposition of the planar graph and the combination of the partial assignments to a near optimal solution require an appropriate planar separator theorem.

Another research direction is the conjecture that, given a graph $G(V, E)$ and a partition of V into κ cliques, G is Hamiltonian iff there exists a Hamiltonian cycle containing at most $2(\kappa - 1)$ inter-clique edges. This would imply a $n^{\mathcal{O}(\kappa)}$ exact algorithm for radio labelling in the complementary graph $\overline{G}$, given a coloring of $\overline{G}$ with κ colors. In this paper, we prove the conjectured bound for graphs and partitions that contain at least one Hamiltonian cycle H, such that any vertex $v \in V$ has degree at most one in the set of inter-clique edges of H.

Additionally, it may be possible to improve the complexity of the algorithm of Theorem 4 to $\mathcal{O}(f(\kappa)p(n))$, where $f(\kappa)$ is a fixed function of κ, e.g. $f(\kappa) = 2^{\kappa^2}$, and $p(n)$ is a fixed polynomial in n of degree not depending on κ, e.g. $p(n) = n$. This is also suggested by Papadimitriou [12].

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