A Selective Tour through Congestion Games*

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Abstract. We give a sketchy and mostly informal overview of research on algorithmic properties of congestion games in the last ten years. We discuss existence of potential functions and pure Nash equilibria in games with weighted players, simple and fast algorithms that reach a pure Nash equilibrium, and efficient approaches to improving the Price of Anarchy.

1 Introduction

Congestion games and their different variants and generalizations provide an elegant model for competitive resource allocation in large-scale telecommunication and transportation networks and have been the subject of intensive research in Algorithmic Game Theory. In an *atomic congestion game*, a finite set of non-cooperative players, each controlling an unsplittable amount of traffic demand, compete over a finite set of resources. All players using a resource experience a delay given by a non-negative and non-decreasing function of the resource's load. Among a given set of resource subsets (or strategies), each player selects one selfishly trying to minimize her *individual delay*, that is the sum of the delays on the resources in the chosen strategy. A natural solution concept is that of a *pure Nash equilibrium*, a configuration where no player can decrease her individual delay by unilaterally switching to a different strategy. In other applications, we consider *non-atomic congestion games* (or *selfish routing games*) where the traffic demand is divided among an infinite number of players, each controlling an infinitesimal amount of traffic. Then, the Nash equilibrium is essentially unique, under mild assumptions on the delay functions, and all players use strategies of equal minimum delay at equilibrium.

The prevailing research questions about algorithmic properties of congestion games have to do either with establishing the existence of pure Nash equilibria and of potential functions for variants and generalizations of atomic games (see e.g., [23,29,30,34,47]), or with bounding the convergence time to a pure Nash equilibrium if the players select their strategies in a selfish and decentralized fashion (see e.g., [1,22,23,26,27,29,47]), or with quantifying and mitigating the

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inefficiency due to the players' selfish behavior using the Price of Anarchy (see e.g., [2,6,11,13,14,17,21,22,24,29,33,41,43,44,49]).

As for several other areas of Theoretical Computer Science, Paul Spirakis has contributed interesting and significant results in all the three directions above. On the occasion of Paul's 60th birthday, I took the opportunity to write this (highly biased and selective) survey on algorithmic properties of congestion games that focuses either on our joint work with Paul (and with a few other dear friends) or on research work that has been directly inspired by Paul's contribution in the area.

It was a sort of an obvious choice for me, since Paul was the person who introduced me to the main research questions about algorithmic properties of congestion games. In fall 2001, when I was a postdoc and Paul was a distinguished visiting scientist at Max-Planck Institut für Informatik, in Saarbrücken, Paul insisted that we should start working together on congestion games on parallel links with linear delays and weighted players (a.k.a. load balancing games). As our first problem, he proposed us to investigate the existence and efficient computation of pure Nash equilibria and the conjecture that the mixed Nash equilibrium with full support (a.k.a. the fully-mixed equilibrium) maximizes the Price of Anarchy for the objective of maximum delay of the players (the latter was motivated by Paul's previous work in [40,44]). [29] was the result of this effort and the beginning of a fruitful and really enjoyable collaboration with Paul (and also with Spyros, Alexis, Vasilis, Thanasis and others) on algorithmic properties of congestion games. Back in Patras, in fall 2003, Paul, Spyros and I started looking at potential functions for congestion games with weighted players and simple algorithms for efficient computation of pure Nash equilibria (the motivation came from [19,46]). What happened next is described in the following pages. *Paul, thank you for everything and happy birthday!*

1.1 Organization

After a formal definition of atomic and non-atomic congestion games and related notions (Section 2), we discuss existence of potential functions for atomic games with weighted players (Section 3). Next, we show how a pure Nash equilibrium can be reached, using simple and natural algorithms, after as many steps as the number of players in series-parallel and extension-parallel networks (Section 4). In the final part, we bound the Price of Anarchy for atomic congestion games on extension-parallel networks (Section 5.1) and discuss how we can use tolls (Section 5.2), Stackelberg routing (Section 5.3) and the Braess paradox (Section 5.4) to improve the Price of Anarchy of congestion games. With the exception of the results about the Braess paradox, which apply to non-atomic congestion games, we mostly focus on algorithmic properties of atomic congestion games.

2 Congestion Games and Nash Equilibria

An *atomic congestion game* consists of a finite set $N = \{1, \ldots, n\}$ of players, a finite set $E = \{e_1, \ldots, e_m\}$ of edges (or resources), a strategy space $\Sigma_i \subseteq 2^E \setminus \{\emptyset\}$

for each player i, and a non-negative and non-decreasing delay function $d_e(x)$ associated with each edge e. A congestion game has *weighted players* if there is a positive weight w_i associated with each player i. Otherwise, the players are unweighted and we have that $w_i = 1$ for each player i. Throughout this survey, we assume that the players are unweighted, unless stated otherwise. A congestion game has *symmetric strategies* if all players share a common strategy space Σ. A congestion game is *symmetric* if it is unweighted and has symmetric strategies. A congestion game is *linear* if every edge e is associated with a linear delay function $d_e(x) = a_e x + b_e$, with $a_e, b_e \geq 0$.

In many parts of this survey, we consider *symmetric network* congestion games. Then, the players' strategies are determined by a directed network $G(V, E)$ with a distinguished origin o and destination t (a.k.a. an $o - t$ network). The common strategy space of the players is the set of (simple) $o - t$ paths in G, denoted $\mathcal{P}$. To be consistent with the definition of strategies as edge subsets, we regard paths as sets of edges. An $o - t$ network is a *parallel-link* network if each path in $\mathcal{P}$ consists of a single edge. Hence, in congestion games on parallel links the players' common strategy space consists of m singleton strategies, one for each edge.

A *configuration* is a tuple $\boldsymbol{s} = (s_1, \ldots, s_n)$ consisting of a strategy $s_i \in \Sigma_i$ for each player i. For every edge e, we let $s_e = |\{i \in N : e \in s_i\}|$ denote the congestion (or load) induced on e by $\boldsymbol{s}$. If the congestion game has weighted players, e's load in $\boldsymbol{s}$ is $s_e = \sum_{i: e \in s_i} w_i$. Given a congestion game on a directed network G, a configuration $\boldsymbol{s}$ is *acyclic* if there is no directed cycle in G with positive load on all its edges. For a configuration $\boldsymbol{s}$ and a path $p \in \mathcal{P}$, we let $s_p^{\min} = \min_{e \in p}\{s_e\}$ denote the minimum load on some edge of p.

Pure Nash Equilibrium. The individual delay (or cost) of player i in the configuration $\boldsymbol{s}$ is $c_i(\boldsymbol{s}) = \sum_{e \in s_i} d_e(s_e)$. A configuration $\boldsymbol{s}$ is a *pure Nash equilibrium* if no player can improve her individual delay by unilaterally changing her strategy. Formally, $\boldsymbol{s}$ is a pure Nash equilibrium if for every player i and all strategies $s'_i \in \Sigma_i$, $c_i(\boldsymbol{s}) \leq c_i(\boldsymbol{s}_{-i}, s'_i)$.

2.1 Price of Anarchy and Price of Stability

We evaluate configurations using the objective of (weighted) *total delay*. The (weighted) total delay $C(\boldsymbol{s})$ of a configuration $\boldsymbol{s}$ in a congestion game (with weighted players) is the (weighted) sum of players' individual delays in $\boldsymbol{s}$, namely

$$C(\boldsymbol{s}) = \sum_{i \in N} w_i c_i(\boldsymbol{s}) = \sum_{e \in E} s_e d_e(s_e) \,.$$

The *optimal configuration*, usually denoted $\boldsymbol{o}$, achieves a minimum (weighted) total delay $C(\boldsymbol{o})$ among all configurations.

The *Price of Anarchy* (PoA) of a congestion game is the maximum ratio $C(\boldsymbol{s})/C(\boldsymbol{o})$ over all pure Nash equilibria $\boldsymbol{s}$ of the game. The *Price of Stability* (PoS) is the minimum ratio $C(\boldsymbol{s})/C(\boldsymbol{o})$ over all pure Nash equilibria $\boldsymbol{s}$ of the game. In words, the Price of Anarchy (resp. the Price of Stability) is equal to

$C(\boldsymbol{s})/C(\boldsymbol{o})$, where $\boldsymbol{s}$ is the pure Nash equilibrium of maximum (resp. minimum) total delay. The Price of Anarchy (resp. the Price of Stability) for a class of congestion games is the maximum PoA (resp. PoS) of any game in this class.

2.2 Potential Functions and Best Responses

A function Φ that assigns a non-negative number $\Phi(\boldsymbol{s})$ to each configuration $\boldsymbol{s}$ is an exact (resp. weighted) *potential function* if when a player i moves from her current strategy s_i to a new strategy $s'_i \in \Sigma_i$, the difference in the potential value equals the difference in the individual delay of player i (resp. times some function of i's weight w_i). Namely, Φ is an exact potential function if

$$\Phi(\boldsymbol{s}_{-i}, s'_i) - \Phi(\boldsymbol{s}) = c_i(\boldsymbol{s}_{-i}, s'_i) - c_i(\boldsymbol{s})\,.$$

If a game admits an (exact or weighted) potential function, its pure Nash equilibria correspond to the local minima of the potential function.

Rosenthal [48] proved that the pure Nash equilibria of an (unweighted) congestion game correspond to the local optima of the following potential function

$$\Phi(\boldsymbol{s}) = \sum_{e \in E} \sum_{k=1}^{s_e} d_e(k)\,.$$

Hence every congestion game admits at least one pure Nash equilibrium (and possibly many of them). For symmetric network congestion games with general delay functions, Fabrikant, Papadimitriou and Talwar [19] proved that the global minimum of the potential function Φ, and thus a pure Nash equilibrium, can be computed in polynomial time by a min-cost flow computation.

A strategy $s_i \in \Sigma_i$ is a *best response* of player i to a configuration $\boldsymbol{s}_{-i}$ of the remaining players if for all strategies $s'_i \in \Sigma_i$, $c_i(\boldsymbol{s}_{-i}, s_i) \leq c_i(\boldsymbol{s}_{-i}, s'_i)$. A strategy $s'_i \in \Sigma_i$ is an *improvement move* of player i in a configuration $\boldsymbol{s}$ if $c_i(\boldsymbol{s}_{-i}, s'_i) < c_i(\boldsymbol{s})$. For a congestion game that admits a potential function, every improvement move decreases the potential value. Therefore, the Nash dynamics, namely, any sequence of improvement moves, converges to a pure Nash equilibrium in a finite number of steps.

2.3 Non-Atomic Congestion Games

In non-atomic congestion games (or *selfish routing games*), the number of players is infinite and each player controls an infinitesimal amount of traffic. Unless stated otherwise, we assume that the traffic rate is $r = 1$. For simplicity and convenience, when we consider non-atomic congestion games, we focus on symmetric games on an $o-t$ network G. Everything else is defined as above, with the important difference that since the number of players is infinite, a configuration $\boldsymbol{s}$ should be regarded now as a *flow* $\boldsymbol{s} = (s_p)_{p \in \mathcal{P}}$ that assigns an amount of traffic $s_p \geq 0$ to each path p so that $\sum_{p \in \mathcal{P}} s_p = r$.

The delay on each path $p \in \mathcal{P}$ in a configuration $\boldsymbol{s}$ is $d_p(\boldsymbol{s}) = \sum_{e \in p} d_e(s_e)$. A configuration $\boldsymbol{s}$ is a *Nash equilibrium* if it routes all traffic on minimum delay paths, i.e., if for every path p with $s_p > 0$, and every path p', $d_p(\boldsymbol{s}) \leq d_{p'}(\boldsymbol{s})$. Hence, in a Nash equilibrium $\boldsymbol{s}$, all players incur the same delay $D(\boldsymbol{s}) = \min_{p: s_p > 0} d_p(f)$ and the total delay is $C(\boldsymbol{s}) = rD(\boldsymbol{s})$.

Since the equivalent of Rosenthal's potential function is convex for non-atomic games, the Nash equilibrium is essentially unique (under mild assumptions on the delay functions). Therefore, the Price of Anarchy and the Price of Stability coincide and are equal to $C(\boldsymbol{s})/C(\boldsymbol{o})$, where $\boldsymbol{s}$ is the Nash equilibrium configuration and $\boldsymbol{o}$ is the configuration of minimum total delay.

3 Potential Functions for Weighted Players

In [46], Monderer and Shapley presented conditions for the acyclicity of Nash dynamics and for the existence of pure Nash equilibria in non-cooperative games. Most of these conditions are naturally associated with potential functions and their generalizations. One of the most interesting results in [46] is that every finite non-cooperative game with an exact potential function is isomorphic to a congestion game. Motivated by [46], we investigated in [30] which classes of congestions games with weighted players admit a potential function and to which extent we can generalize existence of pure Nash equilibria on parallel-link games with weighted players [29]. We proved that linear congestion games with weighted players admit a weighted potential function that naturally generalizes the potential function of Rosenthal. Hence, any sequence of improvement moves converges to a pure Nash equilibrium.

Theorem 1. *([30]) Every linear congestion game with weighted players admits a weighted potential function and thus, a pure Nash equilibrium.*

Proof. The intuition is that Rosenthal's potential can be generalized to weighted players if the order of the players in the sum of their delays does not make any difference (just as in the case of unweighted players). So, any deviating player can be considered as the last player in the sum. This holds for linear delay functions, since their derivative is constant.

Formally, let $\boldsymbol{s}$ be any configuration of a linear congestion game with weighted players. We let

$$U(\boldsymbol{s}) = \sum_{i \in N} w_i \sum_{e \in s_i} (a_e w_i + b_e)$$

be the weighted total delay of the players in $\boldsymbol{s}$, if each player was alone in the game. We also recall that

$$C(\boldsymbol{s}) = \sum_{i \in N} w_i c_i(\boldsymbol{s}) = \sum_{i \in N} w_i \sum_{e \in s_i} (a_e s_e + b_e) = \sum_{e \in E} s_e (a_e s_e + b_e)$$

is the weighted total delay of the players in configuration $\boldsymbol{s}$.

We next show that $\Phi(\boldsymbol{s}) = (C(\boldsymbol{s}) + U(\boldsymbol{s}))/2$ is a weighted potential function (note that for unweighted players and linear delays, $\Phi(\boldsymbol{s})$ becomes Rosenthal's potential). To this end, we let i be some player switching from her strategy s_i in $\boldsymbol{s}$ to a different strategy s'_i and let $\boldsymbol{s}' = (\boldsymbol{s}_{-i}, s'_i)$ be the resulting configuration. We observe that

$$U(\boldsymbol{s}') - U(\boldsymbol{s}) = w_i \sum_{e \in s'_i \setminus s_i} (a_e w_i + b_e) - w_i \sum_{e \in s_i \setminus s'_i} (a_e w_i + b_e)$$

and that

$$C(\boldsymbol{s}') - C(\boldsymbol{s}) = w_i \sum_{e \in s'_i \setminus s_i} [a_e(2s_e + w_i) + b_e] - w_i \sum_{e \in s_i \setminus s'_i} [a_e(2s_e - w_i) + b_e] \,.$$

Using that for all $e \in s'_i \setminus s_i$, $s'_e = s_e + w_i$, that for all $e \in s_i \setminus s'_i$, $s'_e = s_e - w_i$, and that for all $e \in s'_i \cap s_i$, $s'_e = s_e$, we conclude that

$$\begin{aligned}\Phi(\boldsymbol{s}') - \Phi(\boldsymbol{s}) &= (C(\boldsymbol{s}') - C(\boldsymbol{s}) + U(\boldsymbol{s}') - U(\boldsymbol{s}))/2 \\ &= w_i \sum_{e \in s'_i \setminus s_i} [a_e(s_e + w_i) + b_e] - w_i \sum_{e \in s_i \setminus s'_i} (a_e s_e + b_e) \\ &= w_i(c_i(\boldsymbol{s}') - c_i(\boldsymbol{s})) \,.\end{aligned}$$

Therefore, $\Phi(\boldsymbol{s}) = (C(\boldsymbol{s}) + U(\boldsymbol{s}))/2$ is a weighted potential function for linear congestion games with weighted players. □

The potential function of Theorem 1 is versatile and works for several other generalizations of linear congestions games, by appropriately adapting U in each case. It works e.g., for linear congestion games with static coalitions of players [27, Section 6], for linear congestion games in a social context of surplus collaboration [5], and for graphical linear games with weighted players [23].

In [30], we proved that Theorem 1 is essentially best possible, in the sense that (i) congestion games with weighted players and linear delays are not exact potential games, and that (ii) there is a simple congestion game with only two weighted players and delay functions that are either linear or 2-wise linear which admits neither a generalized potential function nor a pure Nash equilibrium.

Shortly after [30], Panagopoulou and Spirakis [47] presented a weighted potential function for congestion games with weighted players and delays given by an exponential function. Subsequently, Harks, Klimm and Möhring [34] significantly strengthened the negative result of [30] by proving that for congestion games with weighted players even the slightest deviation from the settings that guarantee weighted potential functions in [30,47] leads to games that do not admit weighted potentials.

4 Reaching a Pure Nash Equilibrium

The existence of a potential function for congestion games and for linear congestion games with weighted players implies that any sequence of improvement moves converges to a pure Nash equilibrium. Nevertheless, Fabrikant, Papadimitriou and Talwar [19] proved that it is **PLS**-complete to compute a pure Nash equilibrium in symmetric congestion games and in non-symmetric network congestion games. **PLS**-completeness holds even if the delay functions are linear. Moreover, Ackermann, Röglin and Vöcking [1] proved that in symmetric network congestion games with linear delays, where a pure Nash equilibrium can be computed in polynomial time by min-cost flow techniques, there are instances and initial configurations from which any best response sequence is exponentially long. On the positive side, [1] proved that in asymmetric congestion games with general delays, best response dynamics converges fast to a pure Nash equilibrium if (and essentially only if) the strategy space of each player is a matroid.

4.1 Series-Parallel Networks

For symmetric network congestion games, the matroid property corresponds to very simple networks consisting of bunches of parallel links connected in series. Trying to identify some more general classes of symmetric network congestion games where natural and efficient algorithms reach a pure Nash equilibrium, we considered, in [31], series-parallel networks and the so-called Greedy Best Response approach. We recall that an $o-t$ network is *series-parallel* if it consists of either a single edge (o, t) or two series-parallel networks composed either in series or in parallel.

Greedy Best Response, or GBR in brief, considers the players one-by-one in an arbitrary order. Each player adopts her best response strategy given the strategies of the previous players. The choice is irrevocable, in the sense that no player can switch to a different strategy afterwards. We proved that for series-parallel networks, GBR maintains a pure Nash equilibrium. Namely, after a new player selects her strategy, the other players do not have an incentive to deviate.

Theorem 2. *([31]) Greedy Best Response applied to symmetric congestion games on series-parallel networks with general delays maintains a pure Nash equilibrium in time* $O(nm \log m)$.

In [31], we show that for any non-series-parallel network, we can select linear edge delays so that GBR does not maintain a pure Nash equilibrium even for two players. Moreover, we prove that Theorem 2 can be generalized to congestion games with weighted players that satisfy the *common best response* property, namely that all players agree on their best responses with respect to any given collection of edge loads.

4.2 Extension-Parallel Networks

An interesting generalization of congestion games on parallel links is that of symmetric games on extension-parallel networks. An $o-t$ network is *extension-parallel* if it consists of either (i) a single edge (o, t), or (ii) a single edge and an extension-parallel network composed in series, or (iii) two extension-parallel networks composed in parallel. An interesting property of extension-parallel networks is that they have *linearly independent $o-t$ paths*, in the sense that every $o-t$ path contains at least one edge not belonging to any other $o-t$ path (and thus, it is not possible to express a path as the symmetric difference of some other paths, see [35,45]).

In [22], we proved that for symmetric congestion games on extension-parallel networks[1], each player moves at most once in any sequence of best response moves. More formally, we show the following:

Lemma 1. *([22]) For a symmetric congestion game on an extension-parallel network, let $\boldsymbol{s}$ be the current configuration and let i be a player switching from her current strategy s_i to her best response s'_i. Then, for every player j whose current strategy s_j is a best response to $\boldsymbol{s}$, s_j remains a best response of j to the new configuration $\boldsymbol{s}' = (\boldsymbol{s}_{-i}, s'_i)$.*

Lemma 1 directly implies that in extension-parallel networks, the best response dynamics converges to a pure Nash equilibrium in at most n steps. One can also show that the following theorem is essentially best possible, in the sense that it does not hold for any generalization of extension-parallel networks.

Theorem 3. *([22]) For any n-player symmetric congestion game on an extension-parallel network, every sequence of best response moves converges to a pure Nash equilibrium in at most n steps.*

5 The Price of Anarchy and How to Deal with It

Since the seminal paper of Koutsoupias and Papadimitriou [41], the Price of Anarchy of both atomic and non-atomic congestion games has been investigated extensively. Lücking et al. [43] were the first to consider the PoA of atomic congestion games for the objective of total delay. The proved that for parallel-link games with linear delays, the PoA is $4/3$. For parallel-link games with polynomial delays of degree d, Gairing et al. [33] proved that the PoA is at most $d+1$. Awerbuch, Azar and Epstein [6] and Christodoulou and Koutsoupias [13] proved independently that the PoA of congestion games is $5/2$ for linear delays and $d^{\Theta(d)}$ for polynomial delays of degree d. Subsequently, Aland et al. [2] obtained exact bounds on the PoA of congestion games with polynomial delays.

For non-atomic congestion games, Roughgarden [49] proved that the PoA is independent of the strategy space and equal to $\rho(\mathcal{D})$, where ρ depends on the class of delay functions $\mathcal{D}$ only. Specifically, for a non-negative and non-decreasing function $d(x)$,

$$\rho(d) = \sup_{x \geq y \geq 0} \frac{x d(x)}{y d(y) + (x-y) d(x)} \,.$$

[1] Note that matroid congestion games and congestion games on extension-parallel networks have a different combinatorial structure and may have quite different properties. E.g., a network consisting of two parallel-link networks composed in series is not extension-parallel, but corresponds to a symmetric matroid congestion game.

For a non-empty class $\mathcal{D}$ of delay functions, $\rho(\mathcal{D}) = \sup_{d\in\mathcal{D}} \rho(d)$. For example, ρ is equal to 4/3 for linear delays, to $\frac{27+6\sqrt{3}}{23}$ for quadratic delays and to $\Theta(d/\ln d)$ for polynomial delays of degree d. Subsequently, Correa, Schulz, and Stier-Moses [17] introduced the quantities $\beta(d) = 1 - 1/\rho(d)$ and $\beta(\mathcal{D}) = 1 - 1/\rho(\mathcal{D})$, as alternatives to $\rho(d)$ and $\rho(\mathcal{D})$, respectively, and gave a simple and elegant proof of the same bound.

The general picture is that the PoA of atomic congestion games can be quite large and there is a considerable gap between the PoA of atomic and non-atomic congestion games. In fact, for polynomial delays of degree d, this gap is exponential in d. Therefore, it is natural to ask about possible ways of improving the PoA of atomic congestion games either to 1 or at least close to the PoA of non-atomic congestion games. Moreover, it is interesting to investigate possible approaches to further improving the PoA of non-atomic congestion games without expensive changes in the structure of the game.

5.1 The Price of Anarchy for Extension-Parallel Networks

A possible approach to improving the PoA of atomic congestion games is to consider special classes of networks. In contrast to non-atomic games, where the PoA is independent of the strategy space, the PoA of atomic games crucially depends on it (e.g., the PoA of linear congestion games is 4/3 for parallel-links [43] and 5/2 in general [6,13]). In this direction, we [21] and Caragiannis et al. [11] proved independently that the PoA of atomic congestion games on parallel links with delay functions in class $\mathcal{D}$ is at most $\rho(\mathcal{D})$, i.e., it is bounded from above by the PoA of non-atomic congestion games.

Theorem 4. *([11,21]) The Price of Anarchy of atomic congestion games on parallel links with delay functions in class $\mathcal{D}$ is at most $\rho(\mathcal{D})$.*

Proof. We consider a congestion game on a set E of parallel links with delay functions $\{d_e(x)\}_{e\in E} \subseteq \mathcal{D}$. Let $\boldsymbol{o}$ be the optimal configuration, and let $\boldsymbol{s}$ be the pure Nash equilibrium of maximum total delay. For every link $e \in E$,

$$\begin{aligned} s_e d_e(s_e) &= o_e d_e(s_e) + (s_e - o_e) d_e(s_e) \\ &\leq o_e d_e(o_e) + \beta(\mathcal{D}) s_e d_e(s_e) + (s_e - o_e) d_e(s_e)\,, \end{aligned} \tag{1}$$

where the inequality follows from the definitions of $\beta(d)$ and $\beta(\mathcal{D})$.

For every link e with $o_e > s_e$,

$$\begin{aligned} s_e d_e(s_e) &= o_e d_e(o_e) - o_e d_e(o_e) + s_e d_e(s_e) \\ &\leq o_e d_e(o_e) - (o_e - s_e) d_e(s_e + 1)\,. \end{aligned} \tag{2}$$

The inequality follows from $d_e(s_e) \leq d_e(s_e+1)$ and $d_e(s_e+1) \leq d_e(o_e)$, because the delays are non-decreasing and $s_e + 1 \leq o_e$.

Now, let us assume that the following holds:

$$\sum_{e:s_e>o_e} (s_e - o_e) d_e(s_e) \leq \sum_{e:o_e>s_e} (o_e - s_e) d_e(s_e + 1)\,. \tag{3}$$

Then, using (1), for links e with $s_e \geq o_e$, using (2), for links e with $o_e > s_e$, and employing (3), we obtain that:

$$\begin{aligned} C(\boldsymbol{s}) \leq & \sum_{e \in E} o_e d_e(o_e) + \beta(\mathcal{D}) \sum_{e: s_e \geq o_e} s_e d_e(s_e) + \\ & \sum_{e: s_e > o_e} (s_e - o_e) d_e(s_e) - \sum_{e: o_e > s_e} (o_e - s_e) d_e(s_e + 1) \\ \leq & \, C(\boldsymbol{o}) + \beta(\mathcal{D}) C(\boldsymbol{s}) \,. \end{aligned}$$

Therefore, $C(\boldsymbol{s}) \leq (1 - \beta(\mathcal{D}))^{-1} C(\boldsymbol{o}) = \rho(\mathcal{D}) C(\boldsymbol{o})$, i.e., the PoA is at most $\rho(\mathcal{D})$.

For parallel-link games, (3) is an immediate consequence of the pure Nash equilibrium condition. Formally, since $\boldsymbol{s}$ is a pure Nash equilibrium, for every link e with $s_e > o_e$ (which implies that $s_e \geq 1$) and every link e',

$$d_e(s_e) \leq d_{e'}(s_{e'} + 1) \,.$$

Then, (3) follows from the fact that

$$\sum_{e: s_e > o_e} (s_e - o_e) = \sum_{e: o_e > s_e} (o_e - s_e) \,,$$

because in parallel-link networks, $\sum_{e \in E} s_e = \sum_{e \in E} o_e$. □

We observe that in the proof of Theorem 4, the assumption of parallel-link networks is used only to establish (3). Everything else holds for general symmetric congestion games. Therefore, the upper bound of $\rho(\mathcal{D})$ on the PoA (or, more generally, on the inefficiency of a pure Nash equilibrium $\boldsymbol{s}$) holds if the strategy space and the selected configuration $\boldsymbol{s}$ are such that (3) is satisfied. In [22], we observed that if we regard configurations $\boldsymbol{s}$ and $\boldsymbol{o}$ as flows, (3) essentially states that switching from $\boldsymbol{s}$ to $\boldsymbol{o}$ does not increase the value of Rosenthal's potential function. Intuitively, in such cases, one can reduce (3) to the absence of a negative cost cycle in the circulation $\boldsymbol{o} - \boldsymbol{s}$ with Rosenthal's potential as a cost function. Based on this intuition, one can show that for symmetric network congestion games, (3) holds if $\boldsymbol{s}$ is a minimizer of Rosenthal's potential function. Then, we immediately obtain that:

Theorem 5. *([4,22]) For any symmetric network congestion game with delay functions in class $\mathcal{D}$, the Price of Stability is at most $\rho(\mathcal{D})$.*

Moreover, in [35,22], it is shown that if the network is extension-parallel, any pure Nash equilibrium is a minimizer of Rosenthal's potential function. Therefore, the PoA of symmetric congestion games on extension-parallel networks is bounded from above by the PoA of non-atomic congestion games.

Theorem 6. *([22]) For symmetric network congestion games on extension-parallel networks with delay functions in class $\mathcal{D}$, the Price of Anarchy is at most $\rho(\mathcal{D})$.*

In [22], we presented a congestion game with 3 players on a simple series-parallel network with linear delays and PoA equal to $15/11 > 4/3$, i.e., larger than the PoA of non-atomic congestion games with linear delays.

5.2 Optimal Tolls for Atomic Congestion Games

With the PoA of (atomic and non-atomic) congestion games very well understood, a few natural approaches to reducing it have been investigated. A strong approach is to introduce economic incentives, usually modeled as edge-dependent per-unit-of-traffic *tolls*, that influence the players' selfish choices and induce the optimal configuration as a pure Nash equilibrium (and in the ideal case, as the unique pure Nash equilibrium) of the modified game with tolls.

In a modified congestion game with tolls $\boldsymbol{t} = (t_e)_{e \in E}$, the individual cost of a player i in configuration $\boldsymbol{s}$ is equal to $c'_i(\boldsymbol{s}) = \sum_{e \in s_i}(d_e(s_e)+t_e)$, i.e., equal to the total delay through the edges in her strategy s_i plus the tolls for using the edges in s_i. Nash equilibria are now defined with respect to the modified costs $c'_i(\boldsymbol{s})$ that also account for include tolls. However, most of the literature assumes that the tolls are *refundable* to the players and thus, do not affect the social cost. Therefore, each configuration $\boldsymbol{s}$ is evaluated by (and the PoA is defined with respect to) the total delay $C(\boldsymbol{s}) = \sum_{e \in E} s_e d_e(s_e)$ of the players in $\boldsymbol{s}$. The goal in this research direction is to find a set of moderate and efficiently computable *optimal tolls*, under which the Nash equilibria of the modified game coincide with the optimal configuration $\boldsymbol{o}$.

Existence and efficient computation of optimal tolls for non-atomic congestion games have been investigated extensively. A classical result is that the optimal configuration $\boldsymbol{o}$ is realized as the Nash equilibrium of a non-atomic congestion game with *marginal cost tolls* [8]. If the delay functions are differentiable, the marginal cost toll of each edge e is $t_e = o_e d'_e(o_e)$, where $d'_e(x)$ denotes the first derivative of $d_e(x)$. Cole, Dodis, and Roughgarden [16] considered *heterogeneous* players, who may have different valuations of time (delay) in terms of money (toll), and established the existence of optimal tolls for non-atomic symmetric network congestion games through a non-costructive proof based on Brouwer's fixed point theorem. Subsequently, Fleischer, Jain, and Mahdian [20] and Karakostas and Kolliopoulos [37] proved independently that the existence of optimal tolls for non-atomic congestion games with heterogeneous players follows directly from Linear Programming duality. Therefore, optimal tolls can be computed efficiently by solving a Linear Program. These results (and essentially all known results about existence and efficient computation of tolls for non-atomic games) crucially depend on uniqueness of the Nash equilibrium.

For atomic congestion games, that may admit many different pure Nash equilibria, one has to distinguish between the case where a set of tolls *weakly enforces* the optimal configuration $\boldsymbol{o}$, in the sense that $\boldsymbol{o}$ is realized as some pure Nash equilibrium of the modified game with tolls, and the case where a set of tolls *strongly enforces* $\boldsymbol{o}$, in the sense that $\boldsymbol{o}$ is realized as the unique pure Nash equilibrium of the modified game with tolls.

Caragiannis, Kaklamanis, and Kanellopoulos [12] considered atomic congestion games with linear delays and homogeneous players and investigated existence of optimal tolls and how much tolls can improve the Price of Anarchy. They presented a simple non-symmetric congestion game for which the PoA remains at least 6/5 under any set of tolls. Therefore, they proved that non-symmetric congestion games do not necessarily admit strongly optimal tolls. On the positive side, [12] presented (i) a set of strongly optimal tolls for linear congestion games on parallel links, and (ii) efficiently computable tolls that reduce to the PoA to 2 for linear games with arbitrary strategies (and even with weighted players).

Motivated by [12], we investigated in [32] the existence of optimal tolls for symmetric atomic network congestion games with homogeneous players and general delay functions. In [32], we presented a natural toll mechanism, called *cost-balancing tolls*, which are motivated by the optimal tolls for non-atomic games in [20,37]. A set of cost-balancing tolls for a given configuration turns every path with positive load on its edges into a minimum cost path (the optimal tolls for linear games on parallel links in [12] are also based on the same principle). Formally, a set of tolls t is cost-balancing for a configuration $\boldsymbol{s}$ if for every path $p \in \mathcal{P}$ with $s_p^{\min} > 0$ and every path $p' \in \mathcal{P}$,

$$\sum_{e \in p} (d_e(s_e) + t_e) \leq \sum_{e \in p'} (d_e(s_e) + t_e)\,.$$

Essentially by definition, any given configuration $\boldsymbol{s}$ is induced as a pure Nash equilibrium of the modified congestion game with cost-balancing tolls for $\boldsymbol{s}$. We proved that every acyclic configuration $\boldsymbol{s}$ admits cost-balancing tolls. Moreover, the computation of cost-balancing tolls for $\boldsymbol{s}$ naturally reduces to a longest path computation from the origin in the subnetwork used by $\boldsymbol{s}$. Using the fact that the optimal configuration $\boldsymbol{o}$ in symmetric network congestion games is acyclic, we proved the following.

Theorem 7. *([32]) For every symmetric network congestion game, the optimal configuration $\boldsymbol{o}$ is weakly enforceable by cost-balancing tolls $\boldsymbol{t}$ for $\boldsymbol{o}$, which satisfy the following properties:*

(a) Given the optimal configuration $\boldsymbol{o}$, $\boldsymbol{t}$ is computed in time linear in the size of the network.

(b) The maximum toll on any edge is at most $t^{\max} = \delta + \max_{p \in \mathcal{P}} \sum_{e \in p} d_e(n)$, for any $\delta > 0$. Every edge with toll $t^{\max}$ is not used in any pure Nash equilibrium of the modified game with tolls.

(c) The total amount of tolls paid by any player in any pure Nash equilibrium of the modified game with tolls does not exceed $\max_{p: o_p^{\min} > 0} \sum_{e \in p} d_e(o_e)$.

In [32], we gave a simple example where the optimal configuration cannot be weakly enforced by tolls substantially smaller than the cost-balancing tolls of Theorem 7. Therefore, there are symmetric network games where tolls as large as cost-balancing tolls are also necessary for weakly enforcing the optimal configuration. In [28], we generalized Theorem 7 and proved that cost-balancing tolls exist and can be computed efficiently even for heterogeneous players.

The main result of [32] is that for symmetric congestion games on series-parallel networks with increasing delay functions, the optimal configuration is strongly enforceable by the corresponding cost-balancing tolls. Therefore, symmetric congestion games on series-parallel networks with increasing delays admit a set of moderate optimal tolls computable in linear time.

Theorem 8. *([32]) Every symmetric congestion game on a series-parallel network with increasing delay functions admits a set of strongly optimal tolls with the properties (a), (b), and (c) of Theorem 7.*

Interestingly, games on series-parallel networks admit many different pure Nash equilibria in general. However, games on series-parallel networks with cost-balancing tolls admit an essentially unique pure Nash equilibrium that coincides with the optimal configuration!

If the network is not series-parallel, cost-balancing tolls may not strongly enforce the optimal solution even for linear delay functions. Moreover, Theorem 8 cannot be generalized to heterogeneous players. In [28], we presented a simple congestion game on parallel links with linear delay functions and heterogeneous players for which the PoA remains at least 28/27 under any set of tolls.

Given the existence of efficiently computable strongly optimal tolls for congestion games on series-parallel networks, it is natural to ask for optimal tolls that minimize some objective function (e.g. the sum of tolls, the maximum toll, etc.) on the amount of tolls charged to the players. In [32], we proved that even for 2-player linear congestion games on series-parallel networks, it is **NP**-hard to distinguish between the case where the optimal configuration is the unique pure Nash equilibrium (and thus, tolls only serve to increase the players' disutility) and the case where there is another pure Nash equilibrium of total delay at least 6/5 times the optimal total delay (and hence some tolls are required to strongly enforce the optimal configuration).

An intriguing problem that remains open in this research direction is whether strongly optimal tolls exist for symmetric network congestion games with homogeneous players.

5.3 Stackelberg Routing

A different simple and appealing approach to reducing the PoA is *Stackelberg routing* [39]. The idea is to exploit a small fraction of centrally routed (a.k.a. coordinated) players to improve the quality of the Nash equilibrium reached by the remaining selfish players. A *Stackelberg policy* is an algorithm that determines the strategies of the coordinated players. Given the strategies of (and the congestion caused by) the coordinated players, the selfish players lead the system to a configuration where they are at a pure Nash equilibrium. Our goal is to find a Stackelberg policy of minimum Price of Anarchy, that is the worst-case ratio of the total delay of all (coordinated and selfish) players at a Nash equilibrium for the selfish players to the optimal total delay. The PoA of a given Stackelberg strategy is a non-increasing function of the fraction of coordinated players, usually denoted by α, and ideally is given by a continuous curve decreasing from the value of PoA if all players are selfish to 1 if all players are coordinated.

There has been a significant volume of work on the PoA of Stackelberg routing in non-atomic congestion games. For non-atomic linear games on parallel links, Roughgarden [50] proved that it is **NP**-hard to compute an optimal Stackelberg configuration for a given fraction of coordinated players. To deal with

NP-hardness, he proposed two "heuristic" Stackelberg policies, called Scale and Largest Latency First (LLF), and investigated their worst-case PoA as a function of the fraction α of coordinated players. Scale simply employs the optimal configuration scaled by α. LLF assigns the coordinated players to the largest cost strategies in the optimal configuration. Roughgarden proved that the PoA of LLF on parallel links is $1/\alpha$ for general delay functions and $4/(3+\alpha)$ for linear delays.

Swamy [52] and independently Correa and Stier-Moses [18] proved that the PoA of LLF is at most $1+1/\alpha$ for series-parallel networks with general delay functions. Moreover, Swamy proved that the PoA of LLF is at most $\alpha+(1-\alpha)\rho(\mathcal{D})$ for parallel links with delay functions in class $\mathcal{D}$. The best known upper and lower bounds on the PoA of LLF and Scale for non-atomic congestion games on general networks with linear and polynomial delays are due to Karakostas and Kolliopoulos [38]. An upper bound for Scale with linear delays in [38] is $4(1-\alpha^2/4)/3$. Other upper bounds for Scale and the upper bounds for LLF are rather too complicated for stating (and explaining) them in this survey.

In [21], we investigated the PoA of Scale and LLF for atomic congestion games on general networks with linear delays and on parallel-links with general delay functions. We proved that the PoA of LLF is at most $\min\{(20-11\alpha)/8, (3-2\alpha+\sqrt{5-4\alpha})/2\}$ and at least $5(2-\alpha)/(4+\alpha)$. For Scale, we proved that the PoA is at most $\max\{(5-3\alpha)/2, (5-4\alpha)/(3-2\alpha)\}$. These bounds are continuous functions of α and drop from 5/2 to 1, as α grows from 0 to 1. For parallel-link games, we prove that the PoA of LLF matches that for non-atomic games on parallel links, i.e., it is at most $1/\alpha$ for general delays and at most $\alpha+(1-\alpha)\rho(\mathcal{D})$ for delay functions in class $\mathcal{D}$.

The general picture is that for parallel-link networks with general delays and for general networks with polynomial delays, the coordinated players can be allocated so that the PoA decreases smoothly as the fraction α of the coordinated players increases. Unfortunately, there are non-atomic games on $o-t$ networks with delay functions chosen so that the PoA cannot be bounded by any function of α under any Stackelberg configuration [9].

In a different and also very interesting research direction, Kaporis and Spirakis [36] introduced the *price of optimum*, namely the smallest fraction of coordinated players required to induce an optimal configuration. They presented efficient algorithms for computing the price of optimum in Stackelberg routing for non-atomic games on parallel links and on general $o-t$ networks. An interesting consequence of their work is that there are instances where enforcing the optimal configuration may require a large fraction of the coordinated traffic to be sacrificed through slower paths, since optimal configurations can be quite unfair with respect to the players' individual delay.

5.4 Approximate Network Design for Non-Atomic Games

A simple, albeit counterintuitive, way of improving the Price of Anarchy is to exploit the essence of the Braess paradox [10], namely the fact that removing some network edges may improve the players' delay at equilibrium (see Fig. 1

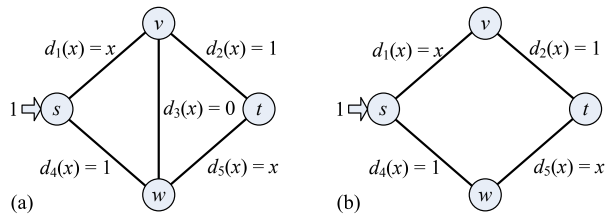

Fig. 1. (a) The optimal total delay is 3/2, achieved by splitting the traffic among the paths (s, v, t) and (s, w, t). In the Nash equilibrium, all traffic goes through the path (s, v, w, t) and has delay 2. This gives a PoA of 4/3. (b) If we remove the edge (v, w), the Nash equilibrium coincides with the optimal configuration. Hence the network on the left is *paradox-ridden*, and the network on the right is its *best subnetwork*.

for a non-atomic congestion game suffering from the paradox). Since Braess's paradox have been studied mostly for non-atomic symmetric network congestion games, we restrict our attention to such games throughout this section.

Focusing on understanding to which extent the PoA can be improved by exploiting the Braess paradox, Roughgarden [51], introduced the optimization problem of the *best subnetwork* (a.k.a. *network design*). Namely, given a non-atomic symmetric network congestion game, to compute the subnetwork induced by edge deletions that minimizes the players' delay at Nash equilibrium (we recall that for non-atomic games, the Nash equilibrium is unique and all players incur the same delay in it). Roughgarden proved that it is **NP**-hard not only to find the best subnetwork, but also to compute any meaningful approximation to the equilibrium delay on the best subnetwork. In particular, he proved that even for linear delays, it is **NP**-hard to distinguish between *paradox-free* instances, where edge removal cannot improve the equilibrium delay, and *paradox-ridden* instances, where the total equilibrium delay on the best subnetwork is equal to the optimal total delay on the original network. Furthermore, Roughgarden proved that for any $\varepsilon > 0$, it is **NP**-hard to approximate the equilibrium delay on the best subnetwork within a factor of $4/3 - \varepsilon$ for linear delays, and within a factor of $\lfloor |V|/2 \rfloor - \varepsilon$ for general delays, where $|V|$ is the number of nodes in the network. Hence, the only general algorithm for approximating the equilibrium delay on the best subnetwork is the trivial one, which does not remove any edges from the network. This algorithm achieves an approximation ratio of 4/3 for linear delays and of $\lfloor |V|/2 \rfloor$ for general delays.

Despite the strong and discouraging results of [51], we proved, in [24], that paradox-ridden instances of the best subnetwork problem can be recognized in polynomial time for networks with strictly increasing linear delay functions. The idea is that if the delay functions are linear and strictly increasing, then the optimal configuration is unique. Therefore, a non-atomic game is paradox-ridden if and only if the unique optimal configuration is a Nash equilibrium for the subnetwork consisting of the edges used by it. In [24], we further generalized this result using properties of Linear Programming and proved the following.

Theorem 9. *([24]) Given a non-atomic symmetric network game with linear delays and at most a constant number of constant delay edges, we can recognize in polynomial time whether it is Braess's-paradox-ridden instance or not.*

If the network is not paradox-ridden, we sought, in [24], for nontrivial special cases that allow for an efficient approximation of the best subnetwork. For networks with polynomially many $o-t$ paths, each of polylogarithmic length, and arbitrary linear delays, we presented a subexponential-time approximation scheme for the equilibrium delay of the best subnetwork. For any $\varepsilon > 0$, the algorithm computes a subnetwork and an ε-Nash equilibrium[2] in it so that the players' delays are within an additive term of $\varepsilon/2$ from the equilibrium delay on the best subnetwork. The running time is exponential in $\mathrm{poly}(\log m)/\varepsilon^2$. The analysis is based on an application of the Probabilistic Method, motivated by Althöfer's Sparsification Lemma [3] and its application to the computation of approximate Nash equilibria for bimatrix games [42]. In particular, we apply the Probabilistic Method and show that any configuration admits an ε-approximate "sparse" configuration that assigns traffic only to $O(\log m/\varepsilon^2)$ paths.

In a subsequent work [25], we presented a subexponential-time approximation for the best subnetwork in sparse random networks. The motivation came from the works of Valiant and Roughgarden [53] and Chung and Young [15], who proved that the Braess paradox occurs with high probability in random $\mathcal{G}_{n,p}$ networks with $p = \Omega(\ln n/n)$, i.e., just greater than the connectivity threshold, and linear delays drawn independently from a natural probability distribution. Our result in [25] is essentially an approximation scheme for a class of so-called *good* instances, which includes the random instances of [15,53] as a special case. Namely, given a good instance and any constant $\varepsilon > 0$, we compute a configuration that (i) is an ε-Nash equilibrium for the subnetwork consisting of the edges used by it, and (ii) has maximum delay no greater than $(1+\varepsilon)D^* + \varepsilon$, where D^* is the equilibrium delay on the best subnetwork.

Our main contribution in [25] is a polynomial-time approximation-preserving reduction of the best subnetwork problem for a good $o-t$ network G to a best subnetwork problem for a 0-*delay simplified network* G_0. The latter is a layered network obtained from G if we keep only o, t and their immediate neighbors, and connect all neighbors of o and t by direct edges of 0 delay. In [25], we proved that the equilibrium delay of the best subnetwork does not increase when we consider the 0-delay simplified network G_0. Although this may sound reasonable, one should be very careful because decreasing edge delays to 0 may trigger the Braess paradox (e.g., starting from the network in Fig. 1.a with $\hat{d}_3(x) = 1$ and decreasing it to $d_3(x) = 0$ is just another way of triggering the paradox). Given the 0-latency simplified network G_0, we can employ the approximation scheme of [24] and approximate the best subnetwork problem on G_0.

The final (and crucial) step of the approximation preserving reduction of [25] is to start with the solution to the best subnetwork problem for the 0-delay

[2] For some $\varepsilon > 0$, a configuration $\boldsymbol{s}$ is an ε-Nash equilibrium if for every path p with $s_p > 0$ and every path p', $d_p(\boldsymbol{s}) \leq d_{p'}(\boldsymbol{s}) + \varepsilon$.

simplified network and extend it to a solution to the best subnetwork problem for the original good network G. In [25], we show how to "simulate" 0-delay edges by low delay paths in the original good network G. Intuitively, this is possible because due to the expansion properties and the random delay functions of G, the intermediate subnetwork of G, connecting the neighbors of o to the neighbors of t, essentially behaves as a complete bipartite network with 0-delay edges. Interestingly, this is also the key step in the approach of [15,53], showing that the Braess paradox occurs in good networks with high probability. Hence, one could say that the reason that the Braess paradox exists in good networks is the very same reason that the paradox can be efficiently approximated.

Since the approximation preserving reduction above runs in polynomial time, we could replace the subexponential-time approximation scheme of [24], for approximating the best subnetwork on the 0-delay simplified network G_0, with an improved approximation scheme based on the generalization of Althöfer's Sparsification Lemma presented in [7]. We believe that this approach could lead to a polynomial-time approximation scheme for many interesting classes of good instances.

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