Maximum Profit Wavelength Assignment in WDM Rings [1]

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1. Introduction

Wavelength Division Multiplexing (WDM) is a dominating technology in contemporary all-optical networking. It allows several connections to be established through the same fiber links, provided that each of the connections uses a different wavelength. A second requirement is that a connection must use the same wavelength from one end to the other in order to avoid the use of wavelength converters which are costly or slow. In practice, the available bandwidth is limited to few dozens, or at most hundreds, wavelengths per fiber and the situation is not expected to change in the near future. It is therefore impossible to serve a large set of communication requests simultaneously. It thus makes sense to consider the problem of satisfying a maximum profit subset of requests, where profits may represent priorities or actual revenues related to requests. In our model, requests are undirected, which corresponds to full-duplex communication. We describe a request by its connection path and its profit, and formulate the problem in graph-theoretic terms as follows:

MAXIMUM PROFIT PATH COLORING PROBLEM (MAXPR-PC)
Input: a graph G, a set of paths $\mathcal{P}$, a profit function $w : \mathcal{P} \to \mathbf{R}$ and a number of available colors k.
Feasible solution: a set of paths $\mathcal{P}' \subseteq \mathcal{P}$ that can be colored with k colors so that no overlapping paths are assigned the same color.
Goal: maximize $\sum_{p \in \mathcal{P}'} w(p)$.

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Here we study MAXPR-PC in rings with undirected requests and we present a 2-approximation algorithm.

While the cardinality version of the problem (MAXPC) has been studied by several researchers [1, 3], MAXPR-PC has been considered in rather few papers [4–6]. Both MAXPR-PC and MAXPC are $\mathcal{NP}$-hard even in simple networks such as rings and trees; this can be shown by an immediate reduction from the corresponding color minimization problem (see e.g. [1]).

MAXPR-PC in chains is also known as the "weighted k-coloring of intervals" problem, which can be solved exactly as shown by Carlisle and Lloyd [4]. In [6] an algorithm based on linear programming and randomized rounding with approximation ratio 1.49 for MAXPR-PC in rings is presented. Let us note here that, although the algorithm in [6] achieves a better approximation ratio, the algorithm presented here is purely combinatorial, therefore faster and easier to implement. Li et al. [5] study a version of MAXPR-PC where requests are not routed in advance, that is, an appropriate routing and coloring is sought. They also assume directed requests and edge capacities that must be obeyed and present a 2-approximation algorithm for rings.

2. Match and Replace for MAXPR-PC

In this section we present an algorithm for MAXPR-PC in rings. MAXPR-PC in chains can be solved exactly in $O(km\ log m))$ time, using algorithm [4].

In our algorithm, we employ a popular technique used for rings, namely to choose an edge e and remove it from a ring. We call this algorithm Match and Replace; details are given in Algorithm 2. We denote the profit of a set of paths $\mathcal{P}$ with $w(P) = \sum_{p \in \mathcal{P}} w(p)$. Given a set of paths $\mathcal{P}$, the set of paths in $\mathcal{P}$ that are colored with the same color i is called the *i-th color class* of $\mathcal{P}$; we use $\mathcal{P}(i)$ to abbreviate this notion.

Theorem 1 *Match and Replace is a 2-approximation algorithm.*

Proof. Let OPT be the value of any optimal solution of the ring instance, OPT_c be the value of any optimal solution of the instance constrained to path set $\mathcal{P}_c$ and OPT_e be the value of any optimal solution of the instance constrained to path set $\mathcal{P}_e$. Recall that

$$OPT \leq OPT_c + OPT_e \ . \tag{1}$$

Let SOL_c be the value of the solution obtained in step 2 of the algorithm (chain subinstance solution), and SOL be the value of the final solution. Clearly,

$$SOL = SOL_c + w'(M) \tag{2}$$

where $w'(M)$ is the sum of the weights of the edges that belong to the matching M computed in step 5. The instance $(G - e, \mathcal{P}_c, w)$ is solved optimally in step 2.

Algorithm 2 Match and Replace

1: Pick an arbitrary separation edge e of the ring. Let $\mathcal{P}_e$ be the set of paths that use edge e and $\mathcal{P}_c = \mathcal{P} \setminus \mathcal{P}_e$.
2: Color the instance $(G-e, \mathcal{P}_c, w)$ optimally, using the Carlisle-Lloyd algorithm for MAXPR-PC in chains.
3: Let $\mathcal{P}_c(i)$ be the i-th color class of $\mathcal{P}_c$, $1 \leq i \leq k$ (note that some color classes may be empty).
4: Construct a weighted bipartite graph $H = (\mathcal{S} \cup \mathcal{P}_e, E)$, with $\mathcal{S} = \{\mathcal{P}_c(i) : i = 1, \ldots, k\}$. For every pair $(\mathcal{P}_c(i), q) \in \mathcal{S} \times \mathcal{P}_e$, define path set $\mathcal{P}_c(i)^q$ such that $\mathcal{P}_c(i)^q \subseteq \mathcal{P}_c(i)$ and $\forall p \in \mathcal{P}_c(i)^q, p$ and q overlap (that is $\mathcal{P}_c(i)^q$ consists of those paths in $\mathcal{P}_c(i)$ that overlap q). If $w(q) - w(\mathcal{P}_c(i)^q) > 0$ then we add edge $(\mathcal{P}_c(i), q)$ to H with weight $w'(\mathcal{P}_c(i), q) = w(q) - w(\mathcal{P}_c(i)^q)$.
5: Find a maximum weight matching M in H.
6: **for each** edge $(\mathcal{P}_c(i), q) \in M$ **do**
7: uncolor all paths in $\mathcal{P}_c(i)^q$ and color path $q \in \mathcal{P}_e$ with color i.
8: **end for**

Therefore, taking also into account Eq. 2 we have that

$$OPT_c = SOL_c \leq SOL \ . \tag{3}$$

Let $\mathcal{S}_M$ be the subset of $\mathcal{S}$ consisting of $\mathcal{P}_c(i)$'s that are matched by M. Similarly, let $\mathcal{P}_{e,M}$ be the paths in $\mathcal{P}_e$ that participate in M. Finally, let K be the set of the k most profitable paths of $\mathcal{P}_e$. We will now show that

$$OPT_e = w(K) \leq SOL \ . \tag{4}$$

For the sake of analysis we will examine a solution SOL' that Match and Replace would have computed if it had chosen a matching M' of a subgraph H' of H in step 5. Bipartite graph H' has the same node set and the same edge weight function as H, but only a subset of the edges of H, namely for every pair $(\mathcal{P}_c(i), q)$: edge $(\mathcal{P}_c(i), q)$ is in H', if $w(q) - w(\mathcal{P}_c(i)) > 0$ and $q \in K$. Let M' be a maximum matching in H', and let $\mathcal{S}_{M'}$ and $\mathcal{P}_{e,M'}$ be defined analogously for M' as for M. Similar to Eq. 2

$$SOL' = SOL_c + w'(M') \ . \tag{5}$$

Note that $SOL_c = w(\mathcal{S})$ and also that $w'(M') = w(\mathcal{P}_{e,M'}) - \sum_{(P(i),q) \in M'} w(\mathcal{P}_c(i)^q)$ $= w(\mathcal{P}_{e,M'}) - \sum_{(\mathcal{P}_c(i),q) \in M'} [w(\mathcal{P}_c(i)) - w(\mathcal{P}_c(i)^{\neg q})] = w(\mathcal{P}_{e,M'}) - w(\mathcal{S}_{M'}) + \sum_{(\mathcal{P}_c(i),q) \in M'} w(\mathcal{P}_c(i)^{\neg q})$, where $\mathcal{P}_c(i)^{\neg q}$ consists of these paths in $\mathcal{P}_c(i)$ that do not overlap with q. Equation 5 may then be rewritten as follows: $SOL' = w(\mathcal{S} \setminus \mathcal{S}_{M'}) + w(\mathcal{P}_{e,M'}) + \sum_{(\mathcal{P}_c(i),q) \in M'} w(\mathcal{P}_c(i)^{\neg q})$. We observe that $\mathcal{P}_{e,M'} \subseteq K$ and therefore $w(\mathcal{P}_{e,M'}) + w(K \setminus \mathcal{P}_{e,M'}) = w(K)$, so the last sum can be expanded in the following way:

$$SOL' = w(\mathcal{S} \setminus \mathcal{S}_{M'}) + w(K) - w(K \setminus \mathcal{P}_{e,M'}) + \sum_{(\mathcal{P}_c(i),q) \in M'} w(\mathcal{P}_c(i)^{\neg q}) \ . \tag{6}$$

Observe also that for any $\mathcal{P}_c(i) \notin \mathcal{S}_{M'}$ and $q \notin \mathcal{P}_{e,M'}$, there must be no edge between them in H', hence $w(\mathcal{P}_c(i)) \geq w(q)$. Moreover, $w(\mathcal{S} \setminus \mathcal{S}_{M'})$ and $w(K \setminus \mathcal{P}_{e,M'})$ are sums with the same number of terms because $|K| = |\mathcal{S}| = k$ and $|\mathcal{S}_{M'}| = |\mathcal{P}_{e,M'}|$. These observations imply that $w(\mathcal{S} \setminus \mathcal{S}_{M'}) - w(K \setminus \mathcal{P}_{e,M}) \geq 0$, therefore Eq. 6 yields $SOL' \geq w(K)$. Since H' is a subgraph of H, M' is a matching also for H, probably not a maximum one, therefore $w'(M) \geq w'(M')$ which implies, from Eq. 2 and 5, that $SOL \geq SOL'$. Combining this last inequality with $SOL' \geq w(K)$ we obtain Eq. 4. By Eq. 3 and 4, SOL is an upper bound on both OPT_e and OPT_c, which together with Eq. 1 gives $SOL \geq \frac{OPT}{2}$.

Computing a solution for the chain subinstance takes $O(kmlogm)$. Graph H has $O(m)$ nodes (we assume that $k < m$) and $O(km)$ edges, therefore maximum weighted matching of H takes $O(m^2(k + logm))$ time. Therefore the total time complexity is $O(m^2(k + logm))$. $\Diamond$

References

[1] Wan, P.J., Liu, L.: Maximal throughput in wavelength-routed optical networks. In: Multichannel Optical Networks: Theory and Practice. Volume 46 of DIMACS Series in Discrete Mathematics and Theoretical Computer Science, AMS (1998) 15–26

[2] Erlebach, T., Jansen, K.: Maximizing the number of connections in optical tree networks. In: Proceedings of ISAAC ’98. LNCS 1533 (1998) 179–188

[3] Nomikos, C., Pagourtzis, A., Zachos, S.: Satisfying a maximum number of pre-routed requests in all-optical rings. Computer Networks **42**(1) (2003) 55–63

[4] Carlisle, M., Lloyd, E.: On the k-coloring of intervals. Discrete Applied Mathematics **59** (1995) 225–235

[5] Li, J., Li, K., Wang, L., Zhao, H.: Maximizing profits of routing in WDM networks. Journal of Combinatorial Optimization **10**(2) (2005) 99–111

[6] Caragiannis, I.: Wavelength management in WDM rings to maximize the number of connections. In: STACS. (2007) 61–72